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GUMARBEK DAUKEEV**

The department of mathematics and
mathematical modeling

MATHEMATICS 2

Methodological Guidelines and Tasks
for carrying out the calculation-graphical works for students of
all educational programs

Almaty 2020

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Methodological Guidelines and Tasks for carrying out the calculation-graphical works contain tasks for three calculation-graphical works (CGW№1, CGW№2, CGW№3) in the sections “Differential and integral calculus of multivariable functions”, “Differential equations”, “Series” of the discipline “Mathematics 2”.

The basic theoretical questions of the program and the solutions of exemplary embodiments are given.

Tables 45, figures 2, bibl.–10

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Introduction

The program for the course "Mathematics 2" is structured in accordance to the current curricula of AUPET. All students study 3 modules for this course, which corresponds to the total number of credits allocated in the curriculum. As a result of studying the discipline, a student must know the basic formulas and methods for differentiating and integrating of multivariable functions, as well as be able to find optimal methods and use the theory of approximation of functions in solving of applied problems.

Methodological Guidelines contain tasks for three calculation-graphical works (CGWs) in the sections "Differential and integral calculus of multivariable functions", "Differential equations", "Series" of the discipline "Mathematics 2".

The basic methodological instructions are given to each part in the form of formulas for solution the tasks of the first level of complexity and solutions of exemplary embodiments are given.

All calculations can be carried out in the software "MathCAD" of any level.

CGW is performed in a separate thin notebook. In the number of each task the second digit indicates the variant.

1 Calculation-graphical work №1. Differential and integral calculus of multivariable functions

Purpose: master the fundamental concepts and methods of the theory of differential and integral calculus of multivariable functions. Get the skills in calculation of derivatives and integrals.

1.1 Theoretical questions

- 1 Multivariable functions. Partial derivatives. Mixed derivatives.
- 2 Tangent plane and normal to the surface.
- 3 Total differential of multivariable functions and its connection with partial derivatives.
- 4 Extreme points of multivariable functions. Necessary and sufficient conditions.
- 5 Double integrals, its basic properties. Calculus of double integrals in Cartesian coordinates.
- 6 Triple integrals, its basic properties. Calculus of triple integrals in Cartesian coordinates.
- 7 Jacobian. Change of variables in multiple integrals.

1.2 Calculated tasks

Task 1. Find for the function $z = f(x,y)$:

a) $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y};$

b) $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y};$

c) make sure that $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y};$

d) $dz, d^2 z.$

1.1	$z = e^{2x^2+y^2}$	1.2	$z = \frac{y}{x^2}$
1.3	$z = x^3 y^6$	1.4	$z = \cos(x^2 y^2 - 5)$
1.5	$z = \sin(x^3 y)$	1.6	$z = (x^2 - 2y)^5$
1.7	$z = (4x - y^3)^2$	1.8	$z = (5x^3 + 2y)^4$
1.9	$z = (2x^3 - y)^7$	1.10	$z = (4x^2 - 5y^3)^5$
1.11	$z = e^{x^2+y^3}$	1.12	$z = (4x + y)^9$
1.13	$z = \cos(x - 5y)$	1.14	$z = \sin(xy)$
1.15	$z = \cos(3x^2 - y^3)$	1.16	$z = (3x + 2y)^9$
1.17	$z = (5x^2 - 3y^4)^2$	1.18	$z = (x^3 - 4y)^7$
1.19	$z = e^{3xy-4}$	1.20	$z = \cos(xy^2)$
1.21	$z = e^{x^2-y^2}$	1.22	$z = e^{x+y^3}$
1.23	$z = \frac{x}{y}$	1.24	$z = \cos(xy^2)$
1.25	$z = \sin(x^2 - y)$	1.26	$z = x^3 y^2$
1.27	$z = (x - y^2)^5$	1.28	$z = (2x + y)^7$
1.29	$z = (x - 3y)^8$	1.30	$z = (3x^2 - 2y^2)^3$

Task 2. Find the direction of the greatest change of function $u(M) = u(x, y, z)$ at the point $M_0(x_0, y_0, z_0)$.

Nº	$u(M)$	M_0	Nº	$u(M)$	M_0
2.1	$x^2 y + y^2 z + z^2 x$	(1,-1,2)	2.16	$\ln(x^3 + y^3 + z + 1)$	(1,3,0)
2.2	$5xy^3 z^2$	(2,1,-1)	2.17	$x - 2y + e^z$	(-4,-5,0)
2.3	$\ln(x^2 + y^2 + z^2)$	(-1,2,1)	2.18	$x^4 - 3xyz$	(2,2,-4)
2.4	$z \cdot e^{x^2+y^2+z^2}$	(0,0,0)	2.19	$3x^2 y^3 z$	(-2,-3,1)
2.5	$\ln(xy + yz + xz)$	(-2,3,-1)	2.20	e^{xy+z^2}	(-5,0,2)
2.6	$\sqrt{1+x^2+y^2+z^2}$	(1,1,1)	2.21	x^{yz}	(3,1,4)
2.7	$x^2 y + xz^2 - 2$	(1,1,1)	2.22	$(x^2 + y^2 + z^2)^3$	(1,2,-1)

2.8	$xe^y + ye^x - z^2$	(3,0,-2)	2.23	$(x+y)^z$	(1,5,0)
2.9	$3xy^2 + z^2 - xyz$	(1,2,2)	2.24	$x^2y + y^2z - 3z$	(0,-2,-1)
2.10	$5x^2yz + xy^2z + yz^2$	(1,1,1)	2.25	$\frac{10}{x^2 + y^2 + z^2 + 1}$	(-1,2,-2)
2.11	$\frac{x}{x^2 + y^2 + z^2}$	(1,2,2)	2.26	$\ln(1 + x^2 - y^2 + z^2)$	(1,1,1)
2.12	$y^2z - 2xyz + z^2$	(3,1,-1)	2.27	$\frac{x}{y} + \frac{y}{z} - \frac{z}{x}$	(-1,1,1)
2.13	$x^2 + y^2 + z^2 - 2xyz$	(1,-1,2)	2.28	$x^3 + xy^2 - 6xyz$	(1,3,-5)
2.14	$\ln(1 + x + y^2 + z^2)$	(1,1,1)	2.29	$\frac{x}{y} - \frac{y}{z} - \frac{x}{z}$	(2,2,2)
2.15	$x^2 + 2y^2 - 4z^2 - 5$	(1,2,1)	2.30	e^{x-yz}	(1,0,3)

Task 3. Find the partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ of the implicitly given function $z=f(x,y)$: $F(x,y,z)=0$.

Nº	$F(x, y, z)$	Nº	$F(x, y, z)$
3.1	$xyz + 3xy^2 - 5yz^2$	3.16	$xy^2 - z + xyz(z - x)$
3.2	$x^2yz - 2xyz^2 + 3xy^2z$	3.17	$(x+y)z^2 + x(y^2 + z^2)$
3.3	$2x^2y^2z - xyz^3 + x^2yz^2$	3.18	$x - yz + xyz - xz^2$
3.4	$x^2(y+z) - yz^2 + 2xyz$	3.19	$x^3y + y^3z + z^3x$
3.5	$(xy - yz)z^2 + xy^2z$	3.20	$xy^2 + yz^2 - zx^2$
3.6	$2xy^2 - 3xyz + 2xz^2$	3.21	$xyz - x^2y - 2xy^2$
3.7	$(y+z)xyz - (x+y-z)yz^2$	3.22	$(x-z)y^2 + (y-x)z^2$
3.8	$3xyz^2 - 4xy^2z + x^2y^2$	3.23	$xy^2z + x^2yz + xyz^2$
3.9	$-x^2y + 2xyz - 3xz^2$	3.24	$(x-yz)x^2 - 2xyz^2$
3.10	$xyz - z^3 + 2xy^2$	3.25	$(x+y)z^2 + (y+z)x^2$
3.11	$zx(y-x) - 2xyz + z^3$	3.26	$(x-yz)z - z(x+y)$
3.12	$x(y^2 + z^2) + 3xyz - z^3$	3.27	$(x+2y)z^2 + 3xyz$
3.13	$(y+z)x^2 - 2xy^2 + xyz$	3.28	$x^2yz + y^2xz + z^2xy$

3.14	$xy(z-y) - 2x^3 + 3xy^2$	3.29	$(x-y)x^2 + y^2z - 2xyz$
3.15	$(x+y)zx^2 + 2xy^2z - xyz^2$	3.30	$x^3y + y^3z + z^3x$

Task 4. Write the equation of the tangent plane and the normal to the surface S at a given point $M_1(x_1, y_1, z_1)$.

Nº	S	$M_1(x_1, y_1, z_1)$
4.1	$x^2 + y^2 + z^2 + 6z - 4x + 8 = 0$	$M_1(2,1,-1)$
4.2	$x^2 - 4y^2 + z^2 = -2xy$	$M_1(-2,1,2)$
4.3	$x^2 + y^2 + z^2 + 6y + 4x = 8$	$M_1(3,2,1)$
4.4	$x^2 + y^2 + z^2 - xy + 3z = 7$	$M_1(5,2,0)$
4.5	$2x^2 - y^2 + z^2 - 4z + y = 13$	$M_1(2,0,1)$
4.6	$x^2 + y^2 + z^2 + 6y + 4z + 4 = 0$	$M_1(-2,4,-1)$
4.7	$x^2 + z^2 + 5yz + 3y = 46$	$M_1(-5,6,8)$
4.8	$x^2 + y^2 - xz - yz + 8 = 0$	$M_1(-3,2,4)$
4.9	$x^2 + y^2 - 2yz - z^2 + y - 2z = 2$	$M_1(8,-5,4)$
4.10	$x^2 + y^2 - z^2 - 2xz + 2x = z$	$M_1(2,11,-11)$
4.11	$z = x^2 + y^2 - 2xy + 2x - y$	$M_1(4,0,-1)$
4.12	$z = -x^2 + y^2 + 6xz - 4y$	$M_1(8,-1,-10)$
4.13	$z = x^2 - y^2 - 2xy - x - 2y$	$M_1(3,0,8)$
4.14	$x^2 + 2y^2 + z^2 + xz - 4y = 13$	$M_1(5,2,3)$
4.15	$4x^2 + 8y^2 - 14z^2 + 4z + 12x - 8y = 9$	$M_1(0,1,4)$
4.16	$y^2 + z^2 + 6x + y - 5z - 8 = 0$	$M_1(5,2,1)$
4.17	$2x^2 - 3y^2 + 2z^2 + 6yz - 4xz + 8x = 0$	$M_1(0,-4,-1)$
4.18	$x^2 - y^2 + z^2 + 6yz - 4z = 14$	$M_1(-2,-1,-1)$
4.19	$x^2 + y^2 - z^2 + xz - 4y + 8x = 15$	$M_1(2,5,-5)$
4.20	$x^2 - y^2 - z^2 + 16y - 4xz + 8x = 15$	$M_1(4,2,-1)$

4.21	$-3x^2 + y^2 + xz - 4yx = 10$	$M_1(5,1,-5)$
4.22	$x^2 + 2y^2 + z^2 + 4xz - 8 = 0$	$M_1(7,1,-7)$
4.23	$x^2 - 3y^2 + z^2 - 4y + 18x = 0$	$M_1(8,0,-8)$
4.24	$x + y^2 - z^2 + 6z - 14x + 6 = 0$	$M_1(2,0,-2)$
4.25	$-5x^2 + y^2 - z^2 + 6xz - 8x + 8z = 0$	$M_1(1,1,-1)$
4.26	$x^2 - y^2 - z^2 + 6xy - 14xz + 11 = 0$	$M_1(5,-5,-5)$
4.27	$z = -2x^2 + y^2 - 4x + 8y$	$M_1(6,6,-5)$
4.28	$y - x^2 + 5y^2 + z^2 + 6xz = 12$	$M_1(7,1,-7)$
4.29	$z = x^2 + 2y^2 - 4xy + 8y$	$M_1(9,5,-9)$
4.30	$z = 2x^2 + y^2 + 4xy - 14y$	$M_1(3,3,-1)$

Task 5. Investigate for extremum of function.

5.1	$z = xy(12 - x - y)$	5.2	$z = (x - 2)^2 + 2y^2 - 10$
5.3	$z = (x - 5)^2 + y^2 + 1$	5.4	$z = 1 + 15x - 2x^2 - xy - 2y^2$
5.5	$z = 2xy - 2x^2 - 4y^2$	5.6	$z = x\sqrt{y} - x^2 - y + 6x + 3$
5.7	$z = 2xy - 5x^2 - 3y^2 + 2$	5.8	$z = x^2 + xy + y^2 - 6x - 9y$
5.9	$z = xy - x^2 - y^2 + 9$	5.10	$z = 2xy - 3x^2 - 2y^2 + 10$
5.11	$z = x^3 + 8y^3 - 6xy + 1$	5.12	$z = y\sqrt{x} - y^2 - x + 6y$
5.13	$z = 2x^3 + 2y^3 - 6xy + 5$	5.14	$z = 3x^3 + 3y^3 - 9xy + 10$
5.15	$z = x^2 + y^2 - xy + x + y$	5.16	$z = x^2 + xy + y^2 - 2x - y$
5.17	$z = (x - 1)^2 + 2y^2$	5.18	$z = y\sqrt{x} - 2y^2 - x + 14y$
5.19	$z = x^2 + 3(y + 2)^2$	5.20	$z = 2(x + y) - x^2 - y^2$
5.21	$z = xy - 3x^2 - 2y^2$	5.22	$z = x^3 + 8y^3 - 6xy + 5$
5.23	$z = x^3 + y^3 - 3xy$	5.24	$z = 1 + 6x - x^2 - xy - y^2$
5.25	$z = xy(6 - x - y)$	5.26	$z = x^2 + xy + y^2 + x - y + 1$
5.27	$z = 6(x - y) - 3x^2 - 3y^2$	5.28	$z = x^2 - xy + y^2 + 9x - 6y + 2$
5.29	$z = 4(x - y) - x^2 - y^2$	5.30	$z = x^3 + y^2 - 6xy - 39x + 18$

Task 6. Check whether the given function $u(x, y, z)$ is a solution of a partial differential equation.

Nº	Equation	$u(x, y, z)$
6.1	$x^2 \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} - y^2 \frac{\partial^2 u}{\partial y^2} = 0$	$u = \frac{y}{x}$

6.2	$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3(x^3 - y^3)$	$u = \ln \frac{x}{y} + x^3 - y^3$
6.3	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	$u = \ln(x^2(y+1)^2)$
6.4	$y \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x} \cdot (1 + y \ln x)$	$u = x^y$
6.5	$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$	$u = \frac{xy}{x+y}$
6.6	$x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} = 0$	$u = e^{xy}$
6.7	$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$	$u = \sin^2(x - ay)$
6.8	$y^2 \frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2} = 0$	$u = y \sqrt{\frac{y}{x}}$
6.9	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$
6.10	$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$	$u = e^{-\cos(x+ay)}$
6.11	$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$	$u = (x-y)(y-z)(z-x)$
6.12	$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$	$u = x \ln \frac{y}{x}$
6.13	$y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = 0$	$u = \ln(x^2 + y^2)$
6.14	$x^2 \frac{\partial u}{\partial x} + xy \frac{\partial u}{\partial y} + y^2 = 0$	$u = \arcsin xy + \frac{y^2}{3x}$
6.15	$x^2 \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xyu = 0$	$u = e^{xy}$

6.16	$\frac{\partial^2 u}{\partial x \partial y} = 0$	$u = \operatorname{arctg} \frac{x+y}{1-xy}$
6.17	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	$u = \ln(x^2 + y^2 + 2x + 1)$
6.18	$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + u = 0$	$u = \frac{2x+3y}{x^2 + y^3}$
6.19	$\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 = 1$	$u = \sqrt{x^2 + y^2 + z^2}$
6.20	$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$	$u = (x^2 + y^2) \operatorname{tg} \frac{x}{y}$
6.21	$9 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	$u = e^{-(x+3y)} \sin(x+3y)$
6.22	$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$	$u = xe^{\frac{y}{x}}$
6.23	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	$u = \operatorname{arctg} \frac{y}{x}$
6.24	$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$	$u = \operatorname{arctg} \frac{y}{x}$
6.25	$\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x^2} = 0$	$u = \ln(x + e^{-y})$
6.26	$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$	$u = \arcsin \frac{x}{x+y}$
6.27	$\frac{1}{x} \frac{\partial u}{\partial x} + \frac{1}{y} \frac{\partial u}{\partial y} = \frac{u}{y^2}$	$u = \frac{y}{(x^2 - y^2)^5}$
6.28	$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{x+y}{x-y}$	$u = \frac{x^2 + y^2}{x-y}$
6.29	$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{2y}{u}$	$u = \sqrt{2xy + y^2}$

6.30	$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$	$u = \ln(x^2 - y^2)$
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Task 7. Find the full derivative of a composite function $u=u(x,y)$, where $x=x(t)$, $y=y(t)$ at the point $t=t_0$.

№	$u=u(x,y)$, $x=x(t)$, $y=y(t)$
7.1	$u = e^{x-2y}$, $x = \sin t$, $y = t^3$, $t_0 = 0$
7.2	$u = \ln(e^x + e^{-y})$, $x = t^2$, $y = t^3$, $t_0 = -1$
7.3	$u = y^x$, $x = \ln(t-1)$, $y = e^{t/2}$, $t_0 = 2$
7.4	$u = e^{y-2x+2}$, $x = \sin t$, $y = e^{t/3}$, $t_0 = \pi/2$
7.5	$u = x^2 e^y$, $x = \cos t^2$, $y = \sin t$, $t_0 = \pi$
7.6	$u = \ln(e^x + e^y)$, $x = t^2$, $y = t^3$, $t_0 = 1$
7.7	$u = x^y$, $x = e^t$, $y = \ln t$, $t_0 = 2$
7.8	$u = e^{y-2x}$, $x = \arctgt^2$, $y = t^3$, $t_0 = 0$
7.9	$u = x^2 e^{-y}$, $x = (t^2 + 2)$, $y = (t-1)^3$, $t_0 = 5$
7.10	$u = \ln(e^{-x} + e^y)$, $x = t^2$, $y = t^3$, $t_0 = -1$
7.11	$u = e^{y-2x-1}$, $x = \cos t$, $y = \arcsin t$, $t_0 = -\pi/2$
7.12	$u = \arcsin(x/y)$, $x = e^{-t}$, $y = t^3$, $t_0 = -\pi$
7.13	$u = \arccos(\frac{2x}{y})$, $x = \sin t$, $y = \cos t^3$, $t_0 = \pi$
7.14	$u = \frac{x^2}{y+1}$, $x = 1-5t^2$, $y = \arctgt$, $t_0 = 0$
7.15	$u = x/y$, $x = e^{2t}$, $y = 2 - e^{2t}$, $t_0 = 0$
7.16	$u = \ln(e^x + e^{-y})$, $x = t^2$, $y = t^3$, $t_0 = -1$
7.17	$u = \sqrt{x + y^2 + 3}$, $x = \ln t^2$, $y = t^3$, $t_0 = e$
7.18	$u = \arcsin(\frac{x^2}{y})$, $x = \sin t$, $y = \cos t$, $t_0 = \pi$

7.19	$u = \frac{2x^3}{y}, x = 1 - 2t, y = \arccos t^3, t_0 = 1$
7.20	$u = \frac{x}{y} - \frac{y}{x}, x = \sin t, y = \cos t, t_0 = \frac{\pi}{4}$
7.21	$u = \sqrt{y + x^2 + 3}, x = \ln t, y = t^2, t_0 = e$
7.22	$u = \arccos\left(\frac{2x}{y}\right), x = \sin t, y = \cos t^3, t_0 = \pi$
7.23	$u = \operatorname{arc tg}\left(\frac{x}{y}\right), x = \sin 2t, y = \cos 3t^3, t_0 = \pi$
7.24	$u = \sqrt{x^2 + y^2 + 3xy}, x = \ln t, y = t^2, t_0 = 1$
7.25	$u = y/x, x = e^{2t}, y = 1 - e^{3t}, t_0 = 2$
7.26	$u = \arccos\left(\frac{2x}{y}\right), x = \sin 2t, y = \cos t, t_0 = \pi$
7.27	$u = \ln(e^{2x} + e^y), x = t^2, y = t^4, t_0 = 1$
7.28	$u = \operatorname{arc tg}(x+y), x = t^2 + 2, y = 4 - t^2, t_0 = 1$
7.29	$u = \sqrt{x^3 - y^2 + 3xy}, x = \ln t, y = t^3, t_0 = 2$
7.30	$u = \operatorname{arc tg}(xy), x = 1 - \sin t, y = e^{3x}, t_0 = \pi/3$

Task 8. Construct the domain D and calculate its area through the double integral.

8.1 $D: y = 3 - 2x - x^2, y = 1 - x$	8.2 $D: y = x^2, x = -1, x = 2$
8.3 $D: y = -\frac{1}{4}x^2 + x + 3, y = 0$	8.4 $D: y = \frac{1}{4}x^3, y - x = 0$
8.5 $D: y = x^2, y = 2x + 8$	8.6 $D: y = 4x, x = 1, x = 9$
8.7 $D: y^2 = x, y = 0, x = 1, x = 4$	8.8 $D: y = -x^2 + 4, y = 0$
8.9 $D: y = x^2, y = x, x = -\frac{1}{2}$	8.10 $D: y = \frac{x^2}{2}, y = 0$
8.11 $D: y = 4 - x^2, y = 0$	8.12 $D: x = 2 - y - y^2, x = 2$
8.13 $D: y = \frac{1}{2}x + 2, y = 0, x = -3, x = 2$	8.14 $D: y = -x, y = 2x - x^2$

8.15	$D: y = x^3, y = 3x$	8.16	$D: y = x^2 + 2, x = 3, x \geq 0, y \geq 0$
8.17	$D: x = 2 - y^2, x = 0$	8.18	$D: y = x^2 - 6x + 5, y = 0$
8.19	$D: y = x, y = 0, x = 1$	8.20	$D: y = 1 - x^2, y \geq 0$
8.21	$D: y = x^2 - 1, y = 3$	8.22	$D: x = y^2, x = 1$
8.23	$D: y^2 = 1 - x, x \geq 0$	8.24	$D: y = 5x, y = x, x = 3$
8.25	$D: y = x^2 - 3x, y = x$	8.26	$D: y = 2 - x, y = x, x \geq 0$
8.27	$D: y = 2 - x^2, y^3 = x^2$	8.28	$D: y = x^2, x = 1, x = 2$
8.29	$D: y = 4 - x^2, y = 3x, x \geq 0, y \geq 0$	8.30	$D: y^2 = 4x, x = 2$

Task 9. Calculate the double integral over a given domain D .

9.1	$\iint_D (x + 3\sqrt{y}) dx dy,$ $D: -2 \leq x \leq 0, 0 \leq y \leq 1$	9.2	$\iint_D (x + y)^2 dx dy,$ $D: -1 \leq x \leq 2, 0 \leq y \leq 1$	9.3	$\iint_D (2x + 1)\sqrt{y} dx dy,$ $D: -2 \leq x \leq 0, 0 \leq y \leq 1$
9.4	$\iint_D (xy^2 + x^3) dx dy,$ $D: 0 \leq x \leq 1, -1 \leq y \leq 0$	9.5	$\iint_D (x + 2)y^2 dx dy,$ $D: -3 \leq x \leq -1, 0 \leq y \leq 2$	9.6	$\iint_D (y + 1)\sqrt{x} dx dy,$ $D: 0 \leq x \leq 4, -1 \leq y \leq 2$
9.7	$\iint_D y(1 - 3x) dx dy,$ $D: -1/3 \leq x \leq 0, 0 \leq y \leq 1$	9.8	$\iint_D (x^3 - 2)\sqrt{y} dx dy,$ $D: 0 \leq x \leq 1, 4 \leq y \leq 9$	9.9	$\iint_D y^3(5 + x) dx dy,$ $D: -1 \leq x \leq 2, 0 \leq y \leq 1$
9.10	$\iint_D (x^2 + y^2) dx dy,$ $D: 0 \leq x \leq 1, -1 \leq y \leq 0$	9.11	$\iint_D (1 + x)\sqrt{y} dx dy,$ $D: 2 \leq x \leq 3, 0 \leq y \leq 1$	9.12	$\iint_D x^3(1 + 2y) dx dy,$ $D: -1 \leq x \leq 1, 0 \leq y \leq 2$
9.13	$\iint_D (3x + y) dx dy,$ $D: 0 \leq x \leq 1, -1 \leq y \leq 1$	9.14	$\iint_D (2\sqrt{x} \cdot y + 1) dx dy,$ $D: 0 \leq x \leq 1, -1 \leq y \leq 1$	9.15	$\iint_D (2 - 3y)\sqrt[3]{x} dx dy,$ $D: 0 \leq x \leq 1, -1 \leq y \leq 0$

9.16 $\iint_D (x^2 y + y^2) dx dy,$ $D : -2 \leq x \leq 1, -1 \leq y \leq 1$	9.17 $\iint_D (1 - 2\sqrt{x}) y dx dy,$ $D : 0 \leq x \leq 9, -2 \leq y \leq 2$	9.18 $\iint_D (x + 2)^2 y dx dy,$ $D : -2 \leq x \leq -1, 0 \leq y \leq 2$
9.19 $\iint_D (1 - 2y) \sqrt{x} dx dy,$ $D : 0 \leq x \leq 1, 0 \leq y \leq 1/2$	9.20 $\iint_D (2 - x) \sqrt{y} dx dy,$ $D : 1 \leq x \leq 2, 0 \leq y \leq 4$	9.21 $\iint_D (1 + y^2 x) dx dy,$ $D : -1 \leq x \leq 1, 0 \leq y \leq 2$
9.22 $\iint_D (x + 3\sqrt{y}) dx dy,$ $D : -1 \leq x \leq 1, 0 \leq y \leq 1$	9.23 $\iint_D (\sqrt{x} + 3x y) dx dy,$ $D : 0 \leq x \leq 1, -1 \leq y \leq 2$	9.24 $\iint_D (x + 2y) dx dy,$ $D : -1 \leq x \leq 1, -1 \leq y \leq 1$
9.25 $\iint_D (xy^2 + 1) dx dy,$ $D : -2 \leq x \leq 2, 0 \leq y \leq 2$	9.26 $\iint_D (1 + 3\sqrt{x}) y dx dy,$ $D : 0 \leq x \leq 1, 0 \leq y \leq 1$	9.27 $\iint_D (2x - 5)^2 y dx dy,$ $D : -2 \leq x \leq 1, -3 \leq y \leq 3$
9.28 $\iint_D (1 + 3x) \sqrt{y} dx dy,$ $D : 0 \leq x \leq 1, 0 \leq y \leq 1/3$	9.29 $\iint_D (5 + x^2) \sqrt{y} dx dy,$ $D : 0 \leq x \leq 1, 0 \leq y \leq 1$	9.30 $\iint_D (1 - 2y) \sqrt{x} dx dy,$ $D : 0 \leq x \leq 1, 0 \leq y \leq 1/2$

Task 10. Calculate the triple integral over a given domain V .

10.1 $\iiint_V (x + y + 4z^2) dx dy dz;$ $V : -1 \leq x \leq 1, 0 \leq y \leq 2, -1 \leq z \leq 1$	10.2 $\iiint_V x^2 y z dx dy dz,$ $V : -1 \leq x \leq 2, 0 \leq y \leq 2, 2 \leq z \leq 3$
10.3 $\iiint_V (2x^2 + 3y + z) dx dy dz,$ $V : 2 \leq x \leq 3, -1 \leq y \leq 2, 0 \leq z \leq 4$	10.4 $\iiint_V (x^2 + y^2 + z^2) dx dy dz;$ $V : 0 \leq x \leq 3, -1 \leq y \leq 2, 0 \leq z \leq 2$
10.5 $\iiint_V x^2 y^2 z dx dy dz;$ $V : -1 \leq x \leq 3, 0 \leq y \leq 2, -2 \leq z \leq 5$	10.6 $\iiint_V (x + y + z) dx dy dz;$ $V : 0 \leq x \leq 3, -1 \leq y \leq 0, 1 \leq z \leq 2$

10.7	$\iiint_V (2x - y^2 - z) dx dy dz;$ $V : 1 \leq x \leq 5, 0 \leq y \leq 2, -1 \leq z \leq 0$	10.8	$\iiint_V 2xy^2 z dx dy dz;$ $V : 0 \leq x \leq 3, -2 \leq y \leq 0, 1 \leq z \leq 2$
10.9	$\iiint_V 5xyz^2 dx dy dz;$ $V : -1 \leq x \leq 0, 2 \leq y \leq 3, 1 \leq z \leq 2$	10.10	$\iiint_V (x^2 + 2y^2 - z) dx dy dz;$ $V : 0 \leq x \leq 1, 0 \leq y \leq 3, -1 \leq z \leq 2$
10.11	$\iiint_V (x + 2yz) dx dy dz;$ $V : -2 \leq x \leq 0, 0 \leq y \leq 1, 0 \leq z \leq 2$	10.12	$\iiint_V (x + yz^2) dx dy dz;$ $V : 0 \leq x \leq 1, 0 \leq y \leq 2, -1 \leq z \leq 3$
10.13	$\iiint_V (xy + 3z) dx dy dz;$ $V : -1 \leq x \leq 1, 0 \leq y \leq 1, 1 \leq z \leq 2$	10.14	$\iiint_V (xy - z^2) dx dy dz;$ $V : 0 \leq x \leq 2, 0 \leq y \leq 1, -1 \leq z \leq 3$
10.15	$\iiint_V (x^3 + yz) dx dy dz;$ $V : -1 \leq x \leq 2, 0 \leq y \leq 1, 0 \leq z \leq 1$	10.16	$\iiint_V (x^3 + y^2 - z) dx dy dz;$ $V : 0 \leq x \leq 2, 1 \leq y \leq 0, 0 \leq z \leq 1$
10.17	$\iiint_V (2x^2 + y - z^3) dx dy dz;$ $V : 0 \leq x \leq 1, -2 \leq y \leq 1, 0 \leq z \leq 1$	10.18	$\iiint_V x^2 yz^2 dx dy dz;$ $V : 0 \leq x \leq 2, 1 \leq y \leq 2, -1 \leq z \leq 0$
10.19	$\iiint_V (x + y - z) dx dy dz;$ $V : 0 \leq x \leq 4, 1 \leq y \leq 3, -1 \leq z \leq 5$	10.20	$\iiint_V (x + 2y + 3z^2) dx dy dz;$ $V : -1 \leq x \leq 2, 0 \leq y \leq 1, 1 \leq z \leq 2$
10.21	$\iiint_V (3x^2 + 2y + z) dx dy dz;$ $V : 0 \leq x \leq 1, 0 \leq y \leq 1, -1 \leq z \leq 3$	10.22	$\iiint_V (xy - z^3) dx dy dz;$ $V : 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3$

<p>10.23</p> $\iiint_V x^3yz \, dx \, dy \, dz;$ $V : 1 \leq x \leq 2, \quad 1 \leq y \leq 3, \quad 0 \leq z \leq 1$	<p>10.24</p> $\iiint_V xy^2z \, dx \, dy \, dz;$ $V : -2 \leq x \leq 1, \quad 0 \leq y \leq 3, \quad 0 \leq z \leq 3$
<p>10.25</p> $\iiint_V xyz^2 \, dx \, dy \, dz;$ $V : 0 \leq x \leq 2, \quad -1 \leq y \leq 0, \quad 0 \leq z \leq 4$	<p>10.26</p> $\iiint_V (x + yz) \, dx \, dy \, dz;$ $V : 0 \leq x \leq 1, \quad -1 \leq y \leq 4, \quad 0 \leq z \leq 2$
<p>10.27</p> $\iiint_V (x + y^2 - z^2) \, dx \, dy \, dz;$ $V : -2 \leq x \leq 0, \quad 1 \leq y \leq 2, \quad 0 \leq z \leq 5$	<p>10.28</p> $\iiint_V (x + y + z^2) \, dx \, dy \, dz;$ $V : -1 \leq x \leq 0, \quad 0 \leq y \leq 1, \quad 2 \leq z \leq 3$
<p>10.29</p> $\iiint_V (x + y^2 - 2z) \, dx \, dy \, dz;$ $V : 1 \leq x \leq 2, \quad -2 \leq y \leq 3, \quad 0 \leq z \leq 1$	<p>10.30</p> $\iiint_V (x - y - z) \, dx \, dy \, dz;$ $V : 0 \leq x \leq 3, \quad 0 \leq y \leq 1, \quad -2 \leq z \leq 1$

Task 11. Change the order of integration.

<p>11.1</p> $\int_0^0 \int_{-1}^0 dy \int_{-\sqrt{-y}}^f f(x, y) dx$	<p>11.2</p> $\int_0^1 dy \int_{-\sqrt{2-y}}^0 f(x, y) dx$	<p>11.3</p> $\int_0^1 dy \int_{-\sqrt{y}}^0 f(x, y) dx$
<p>11.4</p> $\int_0^{\sqrt{2}} dy \int_0^{\sqrt{2-y^2}} f(x, y) dx$	<p>11.5</p> $\int_0^1 dy \int_0^{\sqrt{y}} f(x, y) dx$	<p>11.6</p> $\int_{-\sqrt{2}}^0 dx \int_0^{\sqrt{2-x^2}} f(x, y) dy$
<p>11.7</p> $\int_{-2}^{-\sqrt{3}} dx \int_{-\sqrt{1-x^2}}^0 f(x, y) dy$	<p>11.8</p> $\int_1^e dx \int_{\ln x}^1 f(x, y) dy$	<p>11.9</p> $\int_0^{\sqrt{2}} dx \int_{x^2-1}^1 f(x, y) dy$
<p>11.10</p> $\int_0^1 dy \int_0^{\sqrt[3]{y}} f(x, y) dx$	<p>11.11</p> $\int_1^2 dx \int_0^{2-y} f(x, y) dy$	<p>11.12</p> $\int_{-2}^{-1} dx \int_{-2-x}^1 f(x, y) dy$
<p>11.13</p> $\int_0^1 dy \int_{-\sqrt{y}}^0 f(x, y) dx$	<p>11.14</p> $\int_0^{\sqrt{3}} dy \int_{arctg y}^{\pi/3} f(x, y) dx$	<p>11.15</p> $\int_{-1}^0 dx \int_{\sqrt[3]{x}}^0 f(x, y) dy$

11.16 $\int_{\sqrt{3}}^2 dx \int_{-\sqrt{4-x^2}}^0 f(x, y) dy$	11.17 $\int_{-2}^{-1} dy \int_{-2-y}^0 f(x, y) dx$	11.18 $\int_0^1 dy \int_0^y f(x, y) dx$
11.19 $\int_0^1 dx \int_{x^2}^1 f(x, y) dy$	11.20 $\int_0^{\sqrt{2}} dy \int_0^{\sqrt{2-y^2}} f(x, y) dx$	11.21 $\int_{-\sqrt{2}}^{-1} dy \int_{-\sqrt{2-y^2}}^0 f(x, y) dx$
11.22 $\int_0^1 dx \int_0^{x^3} f(x, y) dy$	11.23 $\int_0^4 dx \int_0^{4-x} f(x, y) dy$	11.24 $\int_{-1}^1 dx \int_{x^2-1}^0 f(x, y) dy$
11.25 $\int_0^1 dx \int_{-\sqrt{1-x}}^0 f(x, y) dy$	11.26 $\int_{-1}^0 dx \int_{-\sqrt{1-x}}^0 f(x, y) dy$	11.27 $\int_{-2}^{-\sqrt{3}} dx \int_{-\sqrt{4-x^2}}^0 f(x, y) dy$
11.28 $\int_0^2 dy \int_{-\sqrt{2-y}}^{-\sqrt{2-y}} f(x, y) dx$	11.29 $\int_{-1}^0 dy \int_{-\sqrt{-2y}}^0 f(x, y) dx$	11.30 $\int_0^{\pi/4} dx \int_0^{\sin x} f(x, y) dy$

Task 12. Calculate the double integral with the transition to the polar coordinates.

12.1 $\iint_D x^2 + y^2 dxdy,$ $D: x^2 + y^2 = 2.$	12.2 $\iint_D x^2 + 2y^2 dxdy,$ $D: x^2 + y^2 = 9.$	12.3 $\iint_D \sqrt{16+x^2+y^2} dxdy,$ $D: x^2 + y^2 = 9.$
12.4 $\iint_D \sqrt{x^2 + y^2} dxdy,$ $D: x^2 + y^2 = 4, x \geq 0.$	12.5 $\iint_D e^{x^2+y^2} dxdy,$ $D: x^2 + y^2 = 1.$	12.6 $\iint_D \sqrt{x^2 + y^2} dxdy,$ $D: x^2 + y^2 = 3, y \geq 0.$
12.7 $\iint_D \sin(x^2 + y^2) dxdy,$ $D: x^2 + y^2 = \pi/2$	12.8 $\iint_D \frac{1}{x^2 + y^2} dxdy,$ $D: 1 \leq x^2 + y^2 \leq 4.$	12.9 $\iint_D e^{3x^2+3y^2} dxdy,$ $D: x^2 + y^2 = 5.$
12.10 $\iint_D \cos(x^2 + y^2) dxdy,$ $D: x^2 + y^2 = \pi/2.$	12.11 $\iint_D (x^2 + y^2) dxdy,$ $D: x^2 + y^2 = 5.$	12.12 $\iint_D \frac{1}{\sqrt{1-x^2-y^2}} dxdy,$ $D: 0 \leq x^2 + y^2 \leq 1/4.$

12.13 $\iint_D (4x^2 + 4y^2) dx dy$, $D : x^2 + y^2 = 2, x \geq 0, y \geq 0$	12.14 $\iint_D \sqrt{x^2 + y^2 + 5} dx dy$, $D : 0 \leq x^2 + y^2 \leq 2$	12.15 $\iint_D \sqrt{x^2 + y^2 - 16} dx dy$, $D : x^2 + y^2 = 20, x \geq 0$
12.16 $\iint_D \frac{x}{\sqrt{x^2 + y^2}} dx dy$, $D : 1 \leq x^2 + y^2 \leq 4.$	12.17 $\iint_D \sin(x^2 + y^2) dx dy$, $D : x^2 + y^2 = \pi/2.$	12.18 $\iint_D \frac{y}{x^2 + y^2} dx dy$, $D : 1 \leq x^2 + y^2 \leq 3$
12.19 $\iint_D y \cos(x^2 + y^2) dx dy$, $D : x^2 + y^2 = 5$	12.20 $\iint_D \sqrt{x^2 + y^2} dx dy$, $D : x^2 + y^2 = 4, x \geq 0.$	12.21 $\iint_D \frac{y}{\sqrt{x^2 + y^2}} dx dy$, $D : x \leq 0, x^2 + y^2 = 1$
12.22 $\iint_D \frac{y}{\sqrt{x^2 + y^2}} dx dy$, $D : 1 \leq x^2 + y^2 \leq 3$	12.23 $\iint_D \frac{1}{\sqrt{1+x^2+y^2}} dx dy$, $D : x^2 + y^2 = 3$	12.24 $\iint_D 3x^2 y^2 dx dy$, $D : x^2 + y^2 = 1, x \geq 0$
12.25 $\iint_D \sqrt{4+x^2+y^2} dx dy$, $D : x^2 + y^2 = 5$	12.26 $\iint_D y \sqrt{x^2 + y^2} dx dy$, $D : x^2 + y^2 = 9$	12.27 $\iint_D \frac{y}{\sqrt{1+x^2+y^2}} dx dy$, $D : x \leq 0, y \leq 0,$ $x^2 + y^2 = 1$
12.28 $\iint_D x \sqrt{x^2 + y^2} dx dy$, $D : x^2 + y^2 = 25, x \geq 0$	12.29 $\iint_D 2x^2 + 3y^2 dx dy$, $D : x^2 + y^2 = 3, x \geq 0, y \geq 0$	12.30 $\iint_D \sqrt{x^2 + y^2} dx dy$, $D : x^2 + y^2 = 7, x \geq 0$

1.3 Solution of an exemplary embodiment

Task 1. Find for the function $z = f(x, y)$:

- a) $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y};$
- b) $\frac{\partial^2 z}{\partial x^2}; \frac{\partial^2 z}{\partial y^2}; \frac{\partial^2 z}{\partial x \partial y};$

c) make sure that $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$;

d) dz, d^2z .

Solution: the function of several arguments $z = f(x, y, \dots, t)$ can be differentiated by each argument, considering all other arguments to be constant. The partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \dots$ obtained in this case are found by the well-known rules for differentiation of one variable function.

Partial derivatives of higher orders $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}, \dots, \frac{\partial^3 z}{\partial x^2 \partial t}, \dots$ are found

by the same rules: partial derivatives of the second order are derivatives of partial derivatives of the first order, partial derivatives of the third order are derivatives of the second order derivatives, etc.

The total differentials of the function $z = f(x, y)$ of the first and second orders are determined by the formulas:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy;$$

$$d^2z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2 \frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2.$$

a) for the function $z = \ln(x^2 + y^2 + 1)$ the first order partial derivatives have the form:

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2 + 1}, \quad \frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2 + 1};$$

$$b) \frac{\partial^2 z}{\partial x^2} = \left(\frac{2x}{x^2 + y^2 + 1} \right)'_x = \frac{2(x^2 + y^2 + 1) - 2x \cdot 2x}{(x^2 + y^2 + 1)^2} = \frac{2(-x^2 + y^2 + 1)}{(x^2 + y^2 + 1)^2};$$

$$\frac{\partial^2 z}{\partial y^2} = \left(\frac{2y}{x^2 + y^2 + 1} \right)'_y = \frac{2(x^2 - y^2 + 1)}{(x^2 + y^2 + 1)^2};$$

$$\frac{\partial^2 z}{\partial x \partial y} = \left(\frac{2x}{x^2 + y^2 + 1} \right)'_y = -\frac{4xy}{(x^2 + y^2 + 1)^2};$$

$$c) \quad \frac{\partial^2 z}{\partial y \partial x} = \left(\frac{2y}{x^2 + y^2 + 1} \right)'_x = -\frac{4xy}{(x^2 + y^2 + 1)^2},$$

thus, really $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y};$

$$d) \quad dz = \frac{2(-x^2 + y^2 + 1)}{(x^2 + y^2 + 1)^2} dx + \frac{2(x^2 - y^2 + 1)}{(x^2 + y^2 + 1)^2} dy = \\ = \frac{2}{(x^2 + y^2 + 1)^2} ((-x^2 + y^2 + 1)dx + (x^2 - y^2 + 1)dy); \\ d^2 z = \frac{2(-x^2 + y^2 + 1)}{(x^2 + y^2 + 1)^2} dx^2 - \frac{8xy}{(x^2 + y^2 + 1)^2} dx dy + \frac{2(x^2 - y^2 + 1)}{(x^2 + y^2 + 1)^2} dy^2.$$

Task 2. Find the direction of the greatest change of function $u(x, y, z) = x^2 - 2xy + 3z^3$ at the point $M_0(0, 2, 1)$.

Solution: the direction of the greatest change of function $u(M) = x^2 - 2xy + 3z^3$ at the point M_0 is given by the vector $\overrightarrow{\text{grad}} u(M_0)$:

$$\overrightarrow{\text{grad}} u(M_0) = u'_x(M_0)\bar{i} + u'_y(M_0)\bar{j} + u'_z(M_0)\bar{k}.$$

Find partial derivatives:

$$u'_x(M_0) = (2x - 2y)|_{M_0} = 2 \cdot 0 - 2 \cdot 2 = -4; \\ u'_y(M_0) = -2x|_{M_0} = -2 \cdot 0 = 0; \\ u'_z(M_0) = 9z^2|_{M_0} = 9 \cdot 1 = 9.$$

Thus, $\overrightarrow{\text{grad}} u(M_0) = -4\bar{i} + 9\bar{k}$.

Task 3. Find the partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ of the implicitly given function $z = f(x, y)$: $F(x, y, z) = xyz + \ln(x + 2y + 3z) = 0$.

Solution: we calculate the partial derivatives for the function $F(x, y, z)$:

$$F'_x = yz + \frac{1}{x+2y+3z}; \quad F'_y = xz + \frac{2}{x+2y+3z}; \quad F'_z = xy + \frac{3}{x+2y+3z}.$$

Further, for an implicitly given function, partial derivatives are substituted into the formulas:

$$z'_x = -\frac{F'_x}{F'_z} = -\frac{yz + \frac{1}{x+2y+3z}}{xy + \frac{3}{x+2y+3z}}; \quad z'_y = -\frac{F'_x}{F'_z} = -\frac{xz + \frac{2}{x+2y+3z}}{xy + \frac{3}{x+2y+3z}}.$$

Task 4. Write the equation of the tangent plane and the normal to the surface $S: F(x, y, z) = \sqrt{x^2 + y^2 + z^2} = 0$ at the given point $M_1(-2, 3, 6)$.

Solution: the equations of the tangent plane and the normal to the surface $F(x, y, z) = 0$ at the given point $M_1(x_1, y_1, z_1)$ have the form:

$$\left. \frac{\partial F}{\partial x} \right|_{M_1} (x - x_1) + \left. \frac{\partial F}{\partial y} \right|_{M_1} (y - y_1) + \left. \frac{\partial F}{\partial z} \right|_{M_1} (z - z_1) = 0;$$

$$\frac{(x - x_1)}{\left. \frac{\partial F}{\partial x} \right|_{M_1}} = \frac{(y - y_1)}{\left. \frac{\partial F}{\partial y} \right|_{M_1}} = \frac{(z - z_1)}{\left. \frac{\partial F}{\partial z} \right|_{M_1}}.$$

Calculate the partial derivatives at the point:

$$F'_x(M_1) = \left. \frac{x}{\sqrt{x^2 + y^2 + z^2}} \right|_{M_1} = \frac{-2}{\sqrt{4+9+36}} = -\frac{2}{7};$$

$$F'_y(M_1) = \left. \frac{y}{\sqrt{x^2 + y^2 + z^2}} \right|_{M_1} = \frac{3}{\sqrt{4+9+36}} = \frac{3}{7};$$

$$F'_z(M_1) = \left. \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right|_{M_1} = \frac{6}{\sqrt{4+9+36}} = \frac{6}{7},$$

substitute them into the equations:

$$\frac{-2}{7}(x+2) + \frac{3}{7}(y-3) + \frac{6}{7}(z-6) = 0;$$

$$\frac{(x+2)}{-2} = \frac{(y-3)}{3} = \frac{(z-6)}{6}.$$

Task 5. Investigate for extremum of function $z = x^3 + 8y^3 - 6xy + 5$.

Solution: function $z = f(x, y)$ has a maximum (minimum) at the point $M_0(x_0, y_0)$, if its value at this point is greater (less) than its values at all sufficiently close points. Maximum or minimum (extremes) can only be at points that lie inside the domain of definition of a function, in which all its first-order partial derivatives are zero or do not exist. Such points are called critical. Not every critical point is an extremum point. To check the critical point for extremum, it is necessary to apply sufficient conditions.

Let the function $z = f(x, y)$ and the critical point $M_0(x_0, y_0)$ be given, we denote:

$$\begin{aligned}\left. \frac{\partial^2 z}{\partial x^2} \right|_{M_0} &= a_{11}; & \left. \frac{\partial^2 z}{\partial x \partial y} \right|_{M_0} &= a_{12}; & \left. \frac{\partial^2 z}{\partial y^2} \right|_{M_0} &= a_{22}; \\ \delta &= \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2.\end{aligned}$$

If $\delta > 0$, $a_{11} > 0$, then $M_0(x_0, y_0)$ is minimum point;
if $\delta > 0$, $a_{11} < 0$, then $M_0(x_0, y_0)$ is maximum point;
if $\delta < 0$, then there is no extremum at the point $M_0(x_0, y_0)$;
if $\delta = 0$, then more research is needed.

Find partial derivatives of functions $z = x^3 + 8y^3 - 6xy + 5$ and critical points in which they are zero or do not exist and which lie inside the domain of function:

$$z'_x = 3x^2 - 6y; z'_y = 24y^2 - 6x.$$

Solve the system $\begin{cases} 3x^2 - 6y = 0 \\ 24y^2 - 6x = 0 \end{cases}$, from which we find two points:

$M_1(0,0)$ and $M_2(1, \frac{1}{2})$. Both points are critical since the function is defined on the whole plane Oxy .

Let's check these points for an extremum on a sufficient basis:

$$\frac{\partial^2 z}{\partial x^2} = 6x; \quad \frac{\partial^2 z}{\partial x \partial y} = -6; \quad \frac{\partial^2 z}{\partial y^2} = 48y.$$

For the point $M_1(0,0)$ we get:

$$\left. \frac{\partial^2 z}{\partial x^2} \right|_{M_1} = a_{11} = 0; \quad \left. \frac{\partial^2 z}{\partial x \partial y} \right|_{M_1} = a_{12} = -6; \quad \left. \frac{\partial^2 z}{\partial y^2} \right|_{M_1} = a_{22} = 0;$$

$$\delta = \begin{vmatrix} 0 & -6 \\ -6 & 0 \end{vmatrix} = -36 < 0,$$

therefore there is no extremum at the point $M_1(0,0)$.

For the point $M_2(1, \frac{1}{2})$ we get:

$$a_{11} = 6, \quad a_{12} = -6, \quad a_{22} = 24, \quad \delta = \begin{vmatrix} 6 & -6 \\ -6 & 24 \end{vmatrix} = 108 > 0,$$

therefore the point $M_2(1, \frac{1}{2})$ is a minimum point and $z_{\min} = z(M_2) = 4$.

Task 6. Check whether the given function $u(x, y) = \cos(x + ay)$ is a solution of a partial differential equation:

$$a^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u.$$

Solution: find partial derivatives:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (\cos(x + ay)) = -\sin(x + ay); \quad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} (\cos(x + ay)) = -a \sin(x + ay);$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} (-\sin(x + ay)) = -\cos(x + ay); \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} (-a \sin(x + ay)) = -a^2 \cos(x + ay).$$

Substitute them into the left side of the differential equation:

$$a^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u(x, y);$$

$$a^2(-\cos(x + ay)) - a^2 \cos(x + ay) = u(x, y),$$

where we get a violation of identity:

$$-2a^2 \cos(x+ay) \neq \cos(x+ay),$$

whence it follows that this function $u(x, y) = \cos(x+ay)$ is not a solution of differential equation $\frac{\partial^2 u}{\partial x^2} + a^2 \frac{\partial^2 u}{\partial y^2} = 0$.

Task 7. Find the full derivative of a composite function $u(x, y) = \operatorname{ctg}(xy)$, $x(t) = \pi(t^2 - 3)$, $y(t) = \frac{1}{t^2}$ at the point $t_0=2$.

Solution: calculate x_0, y_0 and partial derivatives of $u(x, y)$ in them:

$$x_0 = x(2) = \pi(2^2 - 3) = \pi; \quad y_0 = y(2) = \frac{1}{2^2} = \frac{1}{4};$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\operatorname{ctg}(xy)) = -\frac{y}{\sin^2(xy)} \Big|_{(\pi; 0.25)} = -\frac{0.25}{\sin^2(\frac{\pi}{4})} = -0.5;$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\operatorname{ctg}(xy)) = -\frac{x}{\sin^2(xy)} \Big|_{(\pi; 0.25)} = -\frac{\pi}{\sin^2(\frac{\pi}{4})} = -2\pi.$$

Substitute them into the formula of the full derivative of a composite function:

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt};$$

$$\begin{aligned} \frac{du}{dt} \Big|_{t_0} &= \left(-\frac{y}{\sin^2(xy)} \frac{d}{dt}(\pi(t^2 - 3)) - \frac{y}{\sin^2(xy)} \frac{d}{dt}\left(\frac{1}{t^2}\right) \right) \Big|_{t_0} = \\ &= -\frac{y_0}{\sin^2(x_0 y_0)} 2\pi t_0 + \frac{y_0}{\sin^2(x_0 y_0)} \frac{2}{t_0^3} \Big|_{(\pi; 0.25)} = -0.5 \cdot 2\pi \cdot 2 + (-2\pi) \cdot \frac{2}{8}. \end{aligned}$$

Task 8. Construct the domain $D: y=x^2, x=y^2$ and calculate its area through the double integral.

Solution: the domain D is bounded by parabolic lines ($y=x^2, x=y^2$), shown in figure 1.

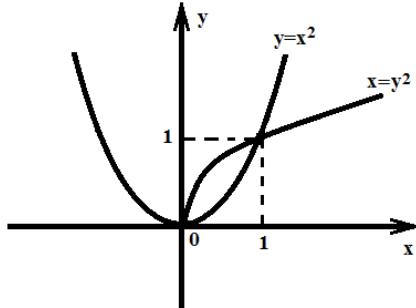


Figure 1

Points of intersection of the parabolas are $M_0(0,0)$ and $M_1(1,1)$. This domain D is “regular” in the direction of the x -axis, we describe it:

$$D : y = x^2, y = \pm\sqrt{x} \Rightarrow D : \begin{cases} 0 \leq x \leq 1 \\ x^2 \leq y \leq \sqrt{x} \end{cases} \Rightarrow$$

$$\iint_D f(x, y) dx dy = \int_0^1 dx \int_{x^2}^{\sqrt{x}} f(x, y) dy.$$

The area of domain D is calculated by double integral:

$$S_D = \iint_D 1 \cdot dx dy = \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy = \int_0^1 \sqrt{x} - x^2 dx = \frac{3}{2} x \sqrt{x} - \frac{1}{3} x^3 \Big|_0^1 = \frac{3}{2} - \frac{1}{3} = \frac{7}{6}.$$

This domain is also “regular” in the direction of the y -axis:

$$D : x = y^2, x = \pm\sqrt{y} \Rightarrow D : \begin{cases} 0 \leq y \leq 1 \\ y^2 \leq x \leq \sqrt{y} \end{cases}.$$

And then it is calculated similarly:

$$S_D = \iint_D 1 \cdot dx dy = \int_0^1 dy \int_{y^2}^{\sqrt{y}} dx = \int_0^1 \sqrt{y} - y^2 dy = \frac{3}{2} y \sqrt{y} - \frac{1}{3} y^3 \Big|_0^1 = \frac{3}{2} - \frac{1}{3} = \frac{7}{6}.$$

Task 9. Calculate the double integral $\iint_D (1+2xy)dxdy$ over a given domain

$$D : 0 \leq x \leq 3, 1 \leq y \leq 2.$$

Solution: since the domain D is “regular”, which is a rectangle, then the limits in the double integral are set as follows:

$$\iint_D (1+2xy)dxdy = \int_0^3 dx \int_1^2 (1+2xy)dy.$$

The inner integral is calculated by integrating with respect to y and assuming that x is constant. Next we integrate with respect to x :

$$\begin{aligned} \int_0^3 dx \int_1^2 (1+2xy)dy &= \int_0^3 (y + xy^2) \Big|_{y=1}^{y=2} dx = \\ &= \int_0^3 ((2+4x)-(1+x))dx = \int_0^3 (1+3x)dx = \left(x + 3\frac{x^2}{2} \right) \Big|_0^3 = \\ &= 3 + \frac{27}{2} = 16\frac{1}{2}. \end{aligned}$$

Task 10. Calculate the triple integral $\iiint_V x + 3y - 2z dxdydz$ over the given domain V , bounded by planes: $x=1, x=3, y=0, y=1, z=0, z=2$.

Solution: since the domain V is “regular”, which is a parallelepiped:

$$V : \begin{cases} 1 \leq x \leq 3 \\ 0 \leq y \leq 1, \\ 0 \leq z \leq 2 \end{cases}$$

then the limits are arranged in order in the triple integral

$$\iiint_V x + 3y - 2z dxdydz = \int_1^3 dx \int_0^1 dy \int_0^2 (x + 3y - 2z) dz.$$

The internal integral is calculated, considering that x and y are constant. Further we integrate with respect to y , assuming that x is constant. And at the end we integrate the resulting function of x with respect to x :

$$\begin{aligned} \int_1^3 dx \int_0^1 dy \int_0^2 (x + 3y - 2z) dz &= \int_1^3 dx \int_0^1 (x + 3y)z - z^2 \Big|_0^2 dy = \\ &= \int_1^3 dx \int_0^1 2(x + 3y) - 4 dy = \int_1^3 2xy + \frac{3}{2}y^2 - 4y \Big|_0^1 dx = \\ &= \int_1^3 2x + \frac{3}{2} - 4 dx = x^2 + \left(\frac{3}{2} - 4\right)x \Big|_0^1 = 1 - \frac{5}{2} = -\frac{3}{2}. \end{aligned}$$

Task 11. Change the order of integration: $\int_0^1 dx \int_{x^2}^x f(x, y) dy$.

Solution: from the double integral it can be seen that the domain of integration is described by the direction of the axis Ox :

$$\int_0^1 dx \int_{x^2}^x f(x, y) dy \Rightarrow D : \begin{cases} 0 \leq x \leq 1 \\ x^2 \leq y \leq x \end{cases} \Leftrightarrow D : \begin{cases} x = 0 \\ x = 1 \\ y = x^2 \\ y = x \end{cases} \Rightarrow D : \begin{cases} x = 0 \\ x = \sqrt{y} \\ x = y \\ y = 0 \text{ if } x = 0 \\ y = 1 \text{ if } x = 1 \end{cases}.$$

Or the domain D is limited by graphs of the same lines, the equations of which have another form ($x = 0$, $x = 1$, $x = \sqrt{y}$, $x = y$), as it shown in figure 2. From here the order of integration in the double integral over the same integration domain changes:

$$\iint_D f(x, y) dx dy = \int_0^1 dy \int_y^{\sqrt{y}} f(x, y) dx.$$

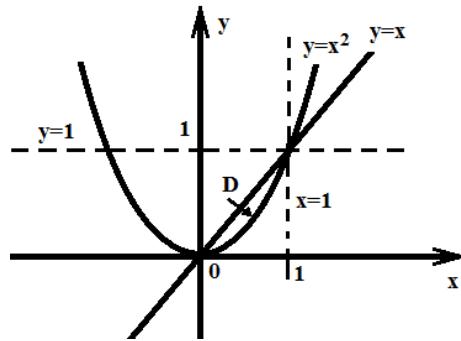


Figure 2

Task 12. Calculate the double integral with the transition to the polar coordinates: $\iint_D \sqrt{2x^2 + y^2} dxdy, \quad D : 1 \leq x^2 + y^2 \leq 4.$

Solution: we describe the domain D in polar coordinates:

$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \end{cases} \Rightarrow x^2 + y^2 = r^2 (\cos^2 \varphi + \sin^2 \varphi) = r^2 \Rightarrow$$

$$D : 1 \leq x^2 + y^2 \leq 4 \Rightarrow D : 1 \leq r^2 \leq 4 \Rightarrow D : \begin{cases} 1 \leq r \leq 2 \\ 0 \leq \varphi \leq 2\pi \end{cases}.$$

Since the domain D is symmetric, we will describe its fourth part D_1 :

$$D = 4D_1 : \begin{cases} 1 \leq r \leq 2 \\ 0 \leq \varphi \leq \frac{\pi}{4} \end{cases}.$$

Next, we make the change of variables in the double integral:

$$\iint_D \frac{x}{\sqrt{x^2 + y^2}} dxdy = \left| \begin{array}{l} x = r \cos \varphi, y = r \sin \varphi \\ dx dy = r dr d\varphi \\ \frac{x}{\sqrt{x^2 + y^2}} = \frac{r \cos \varphi}{r} = \cos \varphi \end{array} \right| = \iint_D \cos \varphi \cdot r dr d\varphi.$$

To calculate the double integral over the domain, we proceed to the iterated integral in polar coordinates:

$$\begin{aligned} \iint_D \cos \varphi \cdot r dr d\varphi &= 4 \iint_{D_1} \cos \varphi \cdot r dr d\varphi = \left| D_1 : \begin{cases} 1 \leq r \leq 2 \\ 0 \leq \varphi \leq \pi/2 \end{cases} \right| = 4 \int_1^2 r dr \int_0^{\pi/2} \cos \varphi d\varphi = \\ &= 4 \frac{r^2}{2} \Big|_1^2 \cdot \sin \varphi \Big|_0^{\pi/2} = 2(4 - 1) \sin \frac{\pi}{2} = 6. \end{aligned}$$

2 Calculation-graphical work №2. Differential equations

Purpose: master the fundamental concepts and methods of the theory of differential equations. Get problem solving skills for differential equations.

2.1 Theoretical questions

- 1 Equations with separable variables. Method for solving these equations.
- 2 Linear inhomogeneous equations. The method of variation of arbitrary constants.
- 3 Equations in total differentials. Solution Method.
- 4 Linear homogeneous differential equations with constant coefficients. Characteristic equation. The structure of the general solution.
- 5 Vronskian. Linear dependence and independence of functions.
- 6 Normal system of differential equations. Characteristic equation for a system of differential equations.
- 7 Numerical series. Convergence and sum of a series. Necessary condition for convergence.
- 8 Series with positive terms. Signs of comparison, d'Alembert's test.
- 9 Series with positive terms. Cauchy's radical and integral tests.
- 10 Power series. Abel's theorem. The radius of convergence. Convergence interval.
- 11 Taylor series. Decomposition of a function in Taylor series.
- 12 Maclaurin series for functions e^x , $\cos x$, $\sin x$.
- 13 Fourier series. Decomposition of functions in Fourier series. Dirichlet theorem.
- 14 Fourier series for functions on an arbitrary interval. The coefficients of the Fourier series.

2.2 Calculated tasks

Task 1. Check whether the specified function is the solution of the differential equation (in this task C is an arbitrary constant).

Nº	Function $y = f(x)$	Equation
1.1	$y = x^2 \left(1 + ce^{\frac{1}{x}} \right)$	$x^2 y' + (1 - 2x)y = x^2$
1.2	$y = xe^{cx}$	$y'x = (1 + \ln y - \ln x)y$
1.3	$y = \frac{cx}{x+1} + 1$	$y - xy' = 1 + x^2 y'$
1.4	$y = (1 + ce^{-e^x})^2$	$y' + 2e^x y = 2e^x \sqrt{y}$
1.5	$y = cx + e^x$	$xy' - y + e^x = 0$
1.6	$y = \frac{2+cx}{1+2x}$	$2(1 + x^2 y') = y - xy'$
1.7	$y = ce^{\frac{\operatorname{tg} \frac{x}{2}}{2}}$	$y' \sin x = y \ln y$
1.8	$y = c\sqrt{x^2 - 1}$	$(x^2 - 1)y' - xy = 0$
1.9	$y = cx^2 e^{-\frac{3}{x}}$	$x^2 y' - 2xy = 3y$
1.10	$y = ce^{-2x} + 2x - 1$	$y' + 2y = 4x$
1.11	$y = cx^2 e^{\frac{1}{x}} + x^2$	$y' + \frac{1-2x}{x^2} y = 1$
1.12	$y = e^{-x^2} \left(c + \frac{x^2}{2} \right)$	$y' + 2xy = xe^{-x^2}$
1.13	$y = (x+c)(1+x^2)$	$(1+x^2)y' - 2xy = (1+x^2)^2$
1.14	$y = cx^2 + \frac{1}{x}$	$x^2 dy + (3 - 2xy)dx = 0$
1.15	$y = ce^{-2x}$	$y' + 2y = 0$
1.16	$y = (x+c)e^x$	$y' - y = e^x$
1.17	$y = c\sqrt{x^2 - 1}$	$(x^2 - 1)y' - xy = 0$
1.18	$y = x^2 (1 + ce^{1/x})$	$x^2 y' + (1 - 2x)y = x^2$
1.19	$y = cx + \frac{1}{c}$	$xy' - y + \frac{1}{y} = 0$
1.20	$y = c \sin x - 2$	$\sin xy' = y \cos x + 2 \cos x$
1.21	$y = \sqrt{x^2 + c}$	$yy' = x$

1.22	$y = \frac{2+cx}{1+2x}$	$2(1+x^2y') = x - xy'$
1.23	$e^{\frac{y}{x}} = cy$	$xyy' - y^2 = x^2y$
1.24	$y = c\sqrt{x^2 - 1}$	$(x^2 - 1)y' - xy = 0$
1.25	$y = \frac{2x}{1-cx^2}$	$y' = \frac{y^2}{x^2} - \frac{y}{x}$
1.26	$y = (c+x)e^x$	$y'' - 2y' + y = 0$
1.27	$y = \cos x + C \cdot \sin x$	$y'' + y = 0$
1.28	$y = \frac{1}{12}(x+C)^3$	$(y'')^2 = y'$
1.29	$y = \frac{1}{3}x^3 + Cx^2$	$y'' = \frac{y'}{x} + x$
1.30	$y = C \cos x + \sin x$	$y'' + y = 0$

Task 2. Find the general solution (general integral) of a differential equation.

№	Task	№	Task
2.1	$xdy - ydx = 0$	2.2	$(x+1)dy + ydx = 0$
2.3	$\sqrt{x}dy - \sqrt{y}dx = 0$	2.4	$(x^2 - 1)dy - dx = 0$
2.5	$x^3dy - \sqrt{x}ydx = 0$	2.6	$\sqrt{1+x}dy - ydx = 0$
2.7	$x^2dy - ydx = 0$	2.8	$xy^2dy + x^3ydx = 0$
2.9	$xydy - dx = 0$	2.10	$xdy - ydx = 0$
2.11	$x\sqrt{y}dy - dx = 0$	2.12	$xdy - ydx = 0$
2.13	$xydy + ydx = 0$	2.14	$xdy - (1-y^2)dx = 0$
2.15	$\sqrt{x}dy - y^2dx = 0$	2.16	$xdy - \sqrt{y^2 + 1}dx = 0$
2.17	$dy - \sqrt{x}ydx = 0$	2.18	$(x+2)^2dy - ydx = 0$
2.19	$xdy - x^2ydx = 0$	2.20	$(1+x^2)dy - dx = 0$
2.21	$ydy - \sqrt{y}dx = 0$	2.22	$\sqrt{1-x^2}dy + ydx = 0$
2.23	$dy - x\sqrt{y}dx = 0$	2.24	$xdy - (1+y^2)dx = 0$
2.25	$y^2dy - xydx = 0$	2.26	$\sqrt{x^2 + 1}dy + ydx = 0$
2.27	$x^3dy - \sqrt{y}dx = 0$	2.28	$ydy - \sqrt{1-y^2}dx = 0$
2.29	$y^3dy + x\sqrt{y}dx = 0$	2.30	$dy - \sqrt{1-y^2}dx = 0$

Task 3. Find the solution of the Cauchy problem and construct the corresponding integral curve.

№	Задание	№	Задание	№	Задание
3.1	$y' = 2y, y(0) = 3$	3.2	$y' = -y, y(0) = 2$	3.3	$y' = y, y(0) = -2$
3.4	$y' = -y, y(0) = -3$	3.5	$y' = 2y, y(0) = -7$	3.6	$y' = -3y, y(0) = -1$
3.7	$y' = 2y, y(0) = 5$	3.8	$y' = -3y, y(0) = 1$	3.9	$y' = \frac{1}{2}y, y(0) = -1$
3.10	$y' = \frac{1}{2}y, y(0) = 4$	3.11	$y' = \frac{1}{3}y, y(0) = 4$	3.12	$y' = 3y, y(-1) = -2$
3.13	$y' = y, y(1) = 2$	3.14	$y' = -y, y(-1) = 2$	3.15	$y' = -y, y(1) = -2$
3.16	$y' = 3y, y(-1) = 2$	3.17	$y' = -y, y(1) = 2$	3.18	$y' = -y, y(-1) = -2$
3.19	$y' = -y, y(3) = -2$	3.20	$y' = y, y(\ln 3) = 4$	3.21	$y' = y, y(1) = -2$
3.22	$y' = k \cdot y, y(0) = y_0$	3.23	$y' = k \cdot y, y(x_0) = y_0$	3.24	$y' = y, y(-3) = 2$
3.25	$y' = -y, y(\ln 2) = -1$	3.26	$y' = \frac{1}{4}y, y(0) = 4$	3.27	$y' = -4y, y(1) = 2 \cdot e^{-4}$
3.28	$y' = -2y, y(1) = 2 \cdot e^{-2}$	3.29	$y' = 2y, y(-\ln 2) = 1$	3.30	$y' = 5y, y(-1) = 2 \cdot e^{-5}$

Task 4. Find the solution of the Cauchy problem.

№	Task	№	Task
4.1	$y' = xy^2, y(0) = 1$	4.2	$y' = (x-3)y^2, y(0) = -1$
4.3	$y' = (x+1)y^2, y(0) = 0$	4.4	$y' = (2x-3)y^2, y(1) = 2$
4.5	$(2-x)y' = y^2, y(-1) = 2$	4.6	$(x+2)y' = y^2, y(1) = -3$
4.7	$y' = (1-x)y^2, y(-1) = -4$	4.8	$(2x+1)y' = -y^2, y(0) = 1$
4.9	$(x+5)y' = -y^2, y(0) = -2$	4.10	$y' = -4xy^2, y(2) = 5$
4.11	$y' = -3xy^2, y(2) = -3$	4.12	$(x+5)y' = -y^2, y(-1) = 4$
4.13	$y' = 2x(1+y^2), y(0) = 3$	4.14	$y'(x+1) = -3y^2, y(0) = -1$
4.15	$y' = xy^3, y(0) = 2$	4.16	$y' = -y^3, y(0) = 4$
4.17	$y' = \frac{x}{y^2}, y(1) = 1$	4.18	$y' = \frac{3x}{y^2}, y(0) = -1$

4.19	$y' = 1 + y^2, y(0) = 1$	4.20	$y' = 4 + y^2, y(1) = \frac{\pi}{2}$
4.21	$y' = y^2 - 1, y(0) = 1$	4.22	$x^2 y' = y^2 - 1, y(1) = 2$
4.23	$xy' = y^2 - 1, y(2) = -2$	4.24	$y' = 1 - y^2, y(0) = -1$
4.25	$y' = 1 - y^2, y(0) = 2$	4.26	$y' = \frac{2xy}{1-y^2}, y(0) = 1$
4.27	$y' = (3+x)y^2, y(0) = 1$	4.28	$y' = \frac{x}{y}, y(0) = -2$
4.29	$y' = \frac{3xy}{4-y^2}, y(0) = 1$	4.30	$y'(x+1) = \frac{3y}{4-y}, y(0) = 1$

Task 5. Find the solution of the Cauchy problem.

№	Task	№	Task
5.1	$y' - \frac{y}{x} = x^3, y(1) = 0$	5.2	$y' - \frac{y}{x} = \sqrt{x}, y(4) = 2$
5.3	$y' - \frac{y}{x} = \frac{1}{\sqrt[3]{x}}, y(1) = 1$	5.4	$y' + \frac{y}{x} = x, y(-1) = 1$
5.5	$y' + \frac{y}{x} = \sqrt{x}, y(4) = -1$	5.6	$y' + \frac{y}{x} = e^x, y(1) = 0$
5.7	$y' + \frac{y}{x} = \frac{2}{1+x^2}, y(1) = 0$	5.8	$y' - 2xy = e^{x^2}, x, y(0) = -1$
5.9	$y' - 2xy = e^{x^2}, y(-1) = 2$	5.10	$y' + 2xy = e^{-x^2}, y(1) = 2$
5.11	$y' + 2xy = x, y(0) = 1$	5.12	$y' + 2xy = 2x, y(0) = -2$
5.13	$y' + \frac{y}{x} = \cos x, y(\pi) = 0$	5.14	$y' - \frac{y}{x} = \ln x, y(e) = 1$
5.15	$y' - \frac{2y}{x} = x, y(1) = 1$	5.16	$y' - \frac{2y}{x} = 1, y(2) = -2$
5.17	$y' - \frac{2y}{x} = \sqrt[3]{x}, y(8) = -1$	5.18	$y' + \frac{2y}{x} = \sqrt{x}, y(4) = 0$
5.19	$y' + \frac{2y}{x} = \frac{\cos 2x}{x^2}, y\left(\frac{\pi}{2}\right) = 0$	5.20	$y' - 2xy = -x, y(0) = -1$
5.21	$y' - \frac{y}{x} = \sqrt[3]{x}, y(8) = 1$	5.22	$y' - \frac{y}{x} = x^2 e^x, y(1) = 1$
5.23	$y' - 2xy = x, y(0) = 1$	5.24	$y' - 2xy = x^2 e^{x^2}, y(1) = 2$
5.25	$y' + 2xy = xe^{-x^2}, y(0) = 2$	5.26	$y' - \frac{y}{x} = -\frac{2}{x^2}, y(1) = 1$

5.27	$y' - y \cos x = \sin 2x, y(0) = -1$	5.28	$y' - 4xy = -4x^3, y(0) = -\frac{1}{2}$
5.29	$y' + \frac{y}{x} = 3x, y(1) = 1$	5.30	$y' + \frac{y}{x} = \sin x, y(\pi) = \frac{1}{\pi}$

Task 6. Find the general solution of the Bernoulli equation.

№	Task	№	Task	№	Task
6.1	$y' + y = xy^2$	6.2	$y' + 2y = xy^3$	6.3	$y' - y = xy^2$
6.4	$y' + y = x\sqrt{y}$	6.5	$y' - y = -x\sqrt{y}$	6.6	$y' - 2y = xy^3$
6.7	$y' + y = x^2y^2$	6.8	$y' + 2y = x^2y^2$	6.9	$y' - y = x^2y^2$
6.10	$y' - y = 2xy^2$	6.11	$y' + y = 2xy^3$	6.12	$y' - 2y = x^2y^2$
6.13	$y' - \frac{1}{3}y = xy^2$	6.14	$y' + 2y = 2xy^2$	6.15	$y' - 2y = \frac{1}{2}xy^2$
6.16	$y' + \frac{1}{x}y = y^2$	6.17	$y' - \frac{2}{x}y = y^2$	6.18	$y' - \frac{1}{x}y = y^2$
6.19	$y' + \frac{2}{x}y = y^2$	6.20	$y' + 3y = xy^2$	6.21	$y' - 3y = xy^3$
6.22	$y' + 2y = y^2e^x$	6.23	$y' + 2y = y^2e^{-2x}$	6.24	$y' + 3y = \frac{x}{y}$
6.25	$y' + 2y = y^2e^{-5x}$	6.26	$y' + 4y = 2\frac{x}{y}$	6.27	$y' - 2y = y^2e^{-x}$
6.28	$y' + \frac{2}{x}y = 2y^2$	6.29	$y' - \frac{1}{4}y = xy^2$	6.30	$y' + 2y = xy^2$

Task 7. Find the general integral of the differential equation in total differentials.

№	Task
7.1	$3x^2e^y dx + (x^3e^y - 1)dy = 0$
7.2	$\left(3x^2 + \frac{2}{y}\cos\frac{2x}{y}\right)dx - \frac{2x}{y^2}\cos\frac{2x}{y}dy = 0$
7.3	$(3x^2 + 4y^2)dx + (8xy + e^y)dy = 0$
7.4	$\left(2x - 1 - \frac{y}{x^2}\right)dx - \left(2y - \frac{1}{x}\right)dy = 0$
7.5	$(y^2 + y \sec^2 x)dx + (2xy + \operatorname{tg} x)dy = 0$

7.6	$(3x^2y + 2y + 3)dx + (x^3 + 2x + 3y^2)dy = 0$
7.7	$\left(\frac{x}{\sqrt{x^2 + y^2}} + \frac{1}{x} + \frac{1}{y} \right)dx + \left(\frac{y}{\sqrt{x^2 + y^2}} + \frac{1}{y} - \frac{x}{y^2} \right)dy = 0$
7.8	$[\sin 2x - 2\cos(x+y)]dx - 2\cos(x+y)dy = 0$
7.9	$(xy^2 + x/y^2)dx + (x^2y - x^2/2y^3)dy = 0$
7.10	$\left(\frac{1}{x^2} + \frac{3y^2}{x^4} \right)dx - \frac{2y}{x^3}dy = 0$
7.11	$\frac{y}{x^2}\cos\frac{y}{x}dx - \left(\frac{1}{x}\cos\frac{y}{x} + 2y \right)dy = 0$
7.12	$\frac{y}{x^2}dx - \frac{xy+1}{x}dy = 0$
7.13	$\frac{1+xy}{x^2}dx + \frac{1-xy}{xy^2}dy = 0$
7.14	$\frac{dx}{y} - \frac{x+y^2}{y^2}dy = 0$
7.15	$\left(\frac{x}{\sqrt{x^2 + y^2}} + y \right)dx + \left(x + \frac{y}{\sqrt{x^2 + y^2}} \right)dy = 0$
7.16	$\left(xe^x + \frac{y}{x^2} \right)dx - \frac{1}{x}dy = 0$
7.17	$\left(10xy - \frac{1}{\sin y} \right)dx + \left(5x^2 + \frac{x \cos y}{\sin^2 y} - y^2 \sin y^3 \right)dy = 0$
7.18	$\left(\frac{y}{x^2 + y^2} + e^x \right)dx - \left(5x^2 + \frac{x \cos y}{\sin^2 y} - y^2 \sin y^3 \right)dy = 0$
7.19	$e^ydx + (\cos y + xe^y)dy = 0$
7.20	$(y^3 + \cos x)dx + (3xy^2 + e^y)dy = 0$
7.21	$xe^{y^2}dx + (x^2y e^{y^2} + \operatorname{tg}^2 y)dy = 0$
7.22	$(5xy^2 - x^3)dx + (5x^2y - y)dy = 0$
7.23	$[\cos(x+y^2) + \sin x]dx + 2y \cos(x+y^2)dy = 0$
7.24	$(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$
7.25	$\left(\sin y + y \sin x + \frac{1}{x} \right)dx + \left(x \cos y - \cos x + \frac{1}{y} \right)dy = 0$

7.26	$\left(1 + \frac{1}{y} e^{x/y}\right)dx + \left(1 - \frac{x}{y^2} e^{x/y}\right)dy = 0$
7.27	$\left(1 + \frac{1}{y} e^x\right)dx + \left(1 - \frac{x}{y^2} e^y\right)dy = 0$
7.28	$2(3xy^2 + 2x^3)dx + 3(2x^2y + y^2)dy = 0$
7.29	$(3x^3 + 6x^2y + 3xy^2)dx + (2x^3 + 3x^2y)dy = 0$
7.30	$xy^2dx + y(x^2 + y^2)dy = 0$

Task 8. Find the general solution of a differential equation.

№	Task	№	Task	№	Task
8.1	$y'' = \sin 2x$	8.2	$y'' = e^{2x}$	8.3	$y'' = xe^x$
8.4	$y'' = \frac{1}{\sqrt[3]{x}}$	8.5	$y'' = \frac{1}{1+x^2}$	8.6	$y'' = \cos 7x$
8.7	$y'' = xe^{-x}$	8.8	$y'' = \frac{\sqrt[3]{x^2-1}}{x}$	8.9	$y'' = \cos^2 x$
8.10	$y'' = \sin^2 2x$	8.11	$y'' = \frac{1}{1+x}$	8.12	$y'' = \frac{x}{(x+1)^3}$
8.13	$y'' = (e^x + e^{-x})^2$	8.14	$y'' = 1 - e^{-\frac{x}{2}}$	8.15	$y'' = 1 - e^{-\frac{x}{2}}$
8.16	$y'' = \sin \frac{x}{2} - \cos \frac{x}{2}$	8.17	$y'' = \frac{1}{\cos^2 x}$	8.18	$y'' = \sqrt{x}(1-x)^2$
8.19	$y'' = \sin x \cdot \cos x$	8.20	$y'' = \frac{1}{\sin^2 x}$	8.21	$y'' = -\frac{1}{1+x^2}$
8.22	$y'' = \frac{1}{\cos^2 2x}$	8.23	$y'' = \cos^2 x$	8.24	$y'' = \frac{1}{2x+1}$
8.25	$y'' = x - \frac{1}{x^3}$	8.26	$y'' = 2^x$	8.27	$y'' = -\sin 4x$
8.28	$y'' = (2+x)^3$	8.29	$y'' = \sin x \cdot \cos^2 x$	8.30	$y'' = 3e^{2x}$

Task 9. Find the general solution of a differential equation.

№	Equation	№	Equation	№	Equation
9.1	$y''x \ln x = y'$	9.2	$xy'' + y' = 1$	9.3	$2xy'' = y'$

9.4	$xy'' + y' = x + 1$	9.5	$\frac{y''}{ctgx} - y' = -\frac{1}{\sin x}$	9.6	$x \cdot y'' - y' = -\frac{1}{x}$
9.7	$y''ctg2x + 2y' = 0$	9.8	$x^3y'' + x^2y' = 1$	9.9	$tgx \cdot y'' = 2y'$
9.10	$y''cth2x = 2y'$	9.11	$x^4y'' + x^3y' = 1$	9.12	$xy'' + 2y' = 0$
9.13	$(1+x^2)y'' + 2xy' = x^3$	9.14	$x^5y'' + x^4y' = 1$	9.15	$x^2y'' + xy' = 1$
9.16	$xy'' + y' + x = 0$	9.17	$y''thx = y'$	9.18	$xy'' + y' = \sqrt{x}$
9.19	$y''tgx = y' + 1$	9.20	$y''tg5x = 5y'$	9.21	$y''th7x = 7y'$
9.22	$x^3y'' + x^2y' = \sqrt{x}$	9.23	$(1+x)y'' + y' = x + 1$	9.24	$(1+\sin x)y'' = \cos x \cdot y'$
9.25	$x \cdot y'' + y' = \frac{1}{\sqrt{x}}$	9.26	$-x \cdot y'' + 2y' = \frac{2}{x^2}$	9.27	$y'' + \frac{2x}{x^2 + 1}y' = 2x$
9.28	$x^4 \cdot y'' + x^3y' = 4$	9.29	$(1+x^2)y'' + 2xy' = 12$	9.30	$cth x \cdot y'' + y' = ch x$

Task 10. Find the solution of the Cauchy problem of homogeneous linear differential equations with constant coefficients.

Nº	Task	Nº	Task
10.1	$y'' - 2y' - 3y = 0,$ $y(0) = 0, y'(0) = 1$	10.2	$y'' + 2y' - 3y = 0,$ $y(0) = 1, y'(0) = 0$
10.3	$y'' - 6y' + 13y = 0,$ $y(0) = 1, y'(0) = -1$	10.4	$y'' + 6y' + 13y = 0,$ $y(0) = -1, y'(0) = 0$
10.5	$y'' + 6y' + 10y = 0,$ $y(0) = 0, y'(0) = -1$	10.6	$y'' - 6y' + 10y = 0,$ $y(0) = -1, y'(0) = 1$
10.7	$y'' + 9y = 0,$ $y(0) = -1, y'(0) = 2$	10.8	$y'' - 2y' + y = 0,$ $y(0) = 1, y'(0) = -1$
10.9	$y'' + y' = 0,$ $y'(0) = 0, y(0) = 1$	10.10	$y'' - 6y' + 8y = 0,$ $y(0) = 2, y'(0) = 1$
10.11	$y'' + 6y' + 8y = 0,$ $y(0) = -1, y'(0) = 2$	10.12	$y'' - 4y' + 4y = 0,$ $y(0) = 1, y'(0) = 1$
10.13	$y'' + 4y' + 4y = 0,$ $y(0) = -2, y'(0) = 0$	10.14	$y'' + 4y' + 5y = 0,$ $y(0) = 0, y'(0) = 1$
10.15	$y'' + 4y' + 8y = 0,$ $y(0) = 0, y'(0) = 1$	10.16	$y'' - 4y' + 5y = 0,$ $y(0) = 1, y'(0) = 0$
10.17	$y'' - 4y' + 8y = 0,$ $y(0) = 0, y'(0) = 1$	10.18	$y'' + 6y' + 9y = 0,$ $y(0) = 1, y'(0) = -1$

10.19	$y''' - y' = 0, y(0) = 0,$ $y'(0) = 0, y''(0) = 2$	10.20	$y'' + 2y' + 10y = 0,$ $y(0) = 0, y'(0) = -1$
10.21	$y'' - 2y' + 10y = 0,$ $y(0) = 1, y'(0) = 1$	10.22	$y'' + 2y' - 8y = 0,$ $y(0) = 1, y'(0) = 2$
10.23	$y'' - 6y' + 9y = 0,$ $y(0) = 2, y'(0) = 1$	10.24	$y'' + 10y' + 26y = 0,$ $y(0) = 0, y'(0) = 1$
10.25	$y'' - 10y' + 26y = 0,$ $y(0) = 1, y'(0) = 0$	10.26	$y'' + 18y + 32 = 0,$ $y(0) = -1, y'(0) = 2$
10.27	$y'' - 4y' + 13y = 0,$ $y(0) = 1, y'(0) = 1$	10.28	$y'' + 10y' + 26y = 0,$ $y(0) = 2, y'(0) = 1$
10.29	$y'' + 4y' - 5y = 0,$ $y(0) = 1, y'(0) = 2$	10.30	$y'' - 6y' + 25y = 0,$ $y(0) = 2, y'(0) = 1$

Task 11. Find the general solution of the differential equation.

№	Task	№	Task
11.1	$y'' + 2y' - 8y = x^2 + 1$	11.2	$y'' - 3y' = x - 2$
11.3	$y'' - 2y' + y = 5e^x$	11.4	$y'' + 2y' - 8y = xe^{4x}$
11.5	$y'' - 2y' - 8y = 7e^{4x}$	11.6	$y'' - 2y' - 8y = -e^{-2x}$
11.7	$y'' + 2y' = x^2 + 2$	11.8	$y'' + 2y' + y = -e^{-x}$
11.9	$y'' - 6y' + 5y = 2x^2 - 5$	11.10	$y'' + 6y' + 5y = x^2 - 3$
11.11	$y'' - 6y' + 8y = 2xe^x$	11.12	$y'' + 6y' + 8y = 5e^{-4x}$
11.13	$y'' + 6y' + 8y = xe^{-2x}$	11.14	$y'' + y = 5\cos x$
11.15	$y'' + y = -\sin x$	11.16	$y'' + y = \cos x - \sin x$
11.17	$y'' - y' = x + x^2 - x^3$	11.18	$y'' + y' = -xe^x$
11.19	$y'' - y' = (1-x)e^{2x}$	11.20	$y'' + y' = (5-x)e^x$
11.21	$y'' - 4y' + 4y = -2e^{2x}$	11.22	$y'' - 4y' + 4y = xe^{-x}$
11.23	$y'' - 4y' + 8y = (1+2x)e^x$	11.24	$y'' + 4y' + 8y = 5 - x^2$
11.25	$y'' + y' + y = (x+2)e^x$	11.26	$y'' - 5y' + 6y = -6x^2 + 2x + 5$
11.27	$y'' - 13y' + 12 = x - 1$	11.28	$y'' - y' = 5x^2 - 1$
11.29	$y'' + y' - 6y = (20x+14)e^{2x}$	11.30	$y'' - 2y' + y = 2x(1-x)$

Task 12. Solve a system of differential equations by the elimination method.

№	Task	№	Task	№	Task
12.1	$\begin{cases} y_1' = -2y_1 + 3y_2, \\ y_2' = 2y_1 - 3y_2 \end{cases}$	12.2	$\begin{cases} y_1' = -y_1 + 2y_2, \\ y_2' = y_1 - 2y_2 \end{cases}$	12.3	$\begin{cases} y_1' = -3y_1 + 2y_2, \\ y_2' = 3y_1 - 2y_2 \end{cases}$
12.4	$\begin{cases} y_1' = -3y_1 + 4y_2, \\ y_2' = 3y_1 - 4y_2 \end{cases}$	12.5	$\begin{cases} y_1' = -4y_1 + 3y_2, \\ y_2' = 4y_1 - 3y_2 \end{cases}$	12.6	$\begin{cases} y_1' = -2y_1 + y_2, \\ y_2' = 2y_1 - y_2 \end{cases}$
12.7	$\begin{cases} y_1' = -5y_1 + y_2, \\ y_2' = 5y_1 - y_2 \end{cases}$	12.8	$\begin{cases} y_1' = -2y_1 + 4y_2, \\ y_2' = 2y_1 - 4y_2 \end{cases}$	12.9	$\begin{cases} y_1' = -y_1 + 4y_2, \\ y_2' = y_1 - 4y_2 \end{cases}$
12.10	$\begin{cases} y_1' = -2y_1 + 5y_2, \\ y_2' = 2y_1 - 5y_2 \end{cases}$	12.11	$\begin{cases} y_1' = -3y_1 + 5y_2, \\ y_2' = 3y_1 - 5y_2 \end{cases}$	12.12	$\begin{cases} y_1' = -y_1 + 5y_2, \\ y_2' = y_1 - 5y_2 \end{cases}$
12.13	$\begin{cases} y_1' = -5y_1 + 2y_2, \\ y_2' = 5y_1 - 2y_2 \end{cases}$	12.14	$\begin{cases} y_1' = -5y_1 + 4y_2, \\ y_2' = 5y_1 - 4y_2 \end{cases}$	12.15	$\begin{cases} y_1' = -5y_1 + 3y_2, \\ y_2' = 5y_1 - 3y_2 \end{cases}$
12.16	$\begin{cases} y_1' = -\frac{1}{2}y_1 + 7y_2, \\ y_2' = \frac{1}{2}y_1 - 7y_2 \end{cases}$	12.17	$\begin{cases} y_1' = -4y_1 + 5y_2, \\ y_2' = 4y_1 - 5y_2 \end{cases}$	12.18	$\begin{cases} y_1' = -2y_1 + 7y_2, \\ y_2' = 2y_1 - 7y_2 \end{cases}$
12.19	$\begin{cases} y_1' = -4y_1 + y_2, \\ y_2' = 4y_1 - y_2 \end{cases}$	12.20	$\begin{cases} y_1' = -y_1 + 7y_2, \\ y_2' = y_1 - 7y_2 \end{cases}$	12.21	$\begin{cases} y_1' = -y_1 + \frac{3}{2}y_2, \\ y_2' = y_1 - \frac{3}{2}y_2 \end{cases}$
12.22	$\begin{cases} y_1' = -y_1 + \frac{5}{2}y_2, \\ y_2' = y_1 - \frac{5}{2}y_2 \end{cases}$	12.23	$\begin{cases} y_1' = -5y_1 + \frac{1}{2}y_2, \\ y_2' = 5y_1 - \frac{1}{2}y_2 \end{cases}$	12.24	$\begin{cases} y_1' = -7y_1 + y_2, \\ y_2' = 7y_1 - y_2 \end{cases}$
12.25	$\begin{cases} y_1' = -0,4y_1 + 2y_2, \\ y_2' = 0,4y_1 - 2y_2 \end{cases}$	12.26	$\begin{cases} y_1' = 0,2y_1 - y_2, \\ y_2' = -0,2y_1 + y_2 \end{cases}$	12.27	$\begin{cases} y_1' = -2,1y_1 + y_2, \\ y_2' = 2,1y_1 - y_2 \end{cases}$
12.28	$\begin{cases} y_1' = -4,5y_1 - 0,5y_2, \\ y_2' = 4,5y_1 + 0,5y_2 \end{cases}$	12.29	$\begin{cases} y_1' = 0,8y_1 - 0,2y_2, \\ y_2' = -0,8y_1 + 0,2y_2 \end{cases}$	12.30	$\begin{cases} y_1' = 3,2y_1 + y_2, \\ y_2' = -3,2y_1 - y_2 \end{cases}$

Task 13. Solve the inhomogeneous differential equation by the method of variation of arbitrary constants.

№	Task
13.1	$y'' + \pi^2 y = \pi^2 / \cos \pi x, y(0) = 3, y'(0) = 0$
13.2	$y'' + 3y' = 9e^{3x} / (1 + e^{3x}), y(0) = \ln 4, y'(0) = 3(1 - \ln 2)$
13.3	$y'' + 4y = 8 \operatorname{ctg} 2x, y\left(\frac{\pi}{4}\right) = 5, y'\left(\frac{\pi}{4}\right) = 4$
13.4	$y'' - 6y' + 8y = 4 / (1 + e^{-2x}), y(0) = 1 + 2 \ln 2, y'(0) = 6 \ln 2$
13.5	$y'' - 9y' + 18y = 9e^{3x} / (1 + e^{-3x}), y(0) = 0, y'(0) = 0$
13.6	$y'' + \pi^2 y = \pi^2 / \sin \pi x, y(1/2) = 1, y'(1/2) = \pi^2 / 2$
13.7	$y'' + \frac{1}{\pi^2} y = \frac{1}{\pi^2 \cos(x/\pi)}, y(0) = 2, y'(0) = 0$
13.8	$y'' - 3y' = \frac{9e^{-3x}}{3 + e^{-3x}}, y(0) = 4 \ln 4, y'(0) = 3(3 \ln 4 - 1)$
13.9	$y'' + y = 4 \operatorname{ctg} x, y(\pi/2) = 4, y'(\pi/2) = 4$
13.10	$y'' - 6y' + 8y = 4 / (2 + e^{-2x}), y(0) = 1 + 3 \ln 3, y'(0) = 10 \ln 3$
13.11	$y'' + 6y + 8y = 4e^{-2x} / (2 + e^{2x}), y(0) = 0, y'(0) = 0$
13.12	$y'' + 9y = 9 / \sin 3x, y(\pi/6) = 4, y'(\pi/6) = 3\pi/2$
13.13	$y'' + 9y = 9 / \cos 3x, y(0) = 1, y'(0) = 0$
13.14	$y'' - y' = e^{-x} / (2 + e^{-x}), y(0) = \ln 27, y'(0) = \ln 9 - 1$
13.15	$y'' + 4y = 4 \operatorname{ctg} 2x, y(\pi/4) = 3, y'(\pi/4) = 2$
13.16	$y'' - 3y' + 2y = \frac{1}{3 + e^{-x}}, y(0) = 1 + 8 \ln 2, y'(0) = 14 \ln 2$
13.17	$y'' - 6y' + 8y = 4e^{2x} / (1 + e^{-2x}), y(0) = 0, y'(0) = 0$
13.18	$y'' + 16y = 16 / \sin 4x, y(\pi/8) = 3, y'(\pi/8) = 2\pi$
13.19	$y'' + 16y = 16 / \cos 4x, y(0) = 3, y'(0) = 0$
13.20	$y'' - 2y' = 4e^{-2x} / (1 + e^{-2x}), y(0) = \ln 4, y'(0) = \ln 4 - 2$
13.21	$y'' + \frac{y}{4} = \frac{1}{4} \operatorname{ctg}(x/2), y(\pi) = 2, y'(\pi) = 1/2$
13.22	$y'' - 3y' + 2y = 1 / (2 + e^{-x}), y(0) = 1 + 3 \ln 3, y'(0) = 5 \ln 3$
13.23	$y'' + 3y' + 2y = e^{-x} / (2 + e^x), y(0) = 0, y'(0) = 0$

13.24	$y'' + 4y = 4/\sin 2x, y(\pi/4) = 2, y'(\pi/4) = \pi$
13.25	$y'' + 4y = 4/\cos 2x, y(0) = 2, y'(0) = 0$
13.26	$y'' + y' = e^x / (2 + e^x), y(0) = \ln 27, y'(0) = 1 - \ln 9$
13.27	$y'' + y = 2\operatorname{ctg} x, y(\pi/2) = 1, y'(\pi/2) = 2$
13.28	$y'' - 3y' + 2y = 1/(1 + e^{-x}), y(0) = 1 + 2\ln 2, y'(0) = 3\ln 2$
13.29	$y'' - 3y' + 2y = e^x / (1 + e^{-x}), y(0) = 0, y'(0) = 0$
13.30	$y'' + y = 1/\sin x, y\left(\frac{\pi}{2}\right) = 1, y'(\pi/2) = \pi/2$

Task 14. Find the general solution of the differential equation.

№	Task	№	Task
14.1	$y''' + 3y'' + 2y' = 1 - x^2$	14.2	$y''' - y' = 6x^2 + 3x$
14.3	$y''' - y' = x^2 + x$	14.4	$y^{IV} + 3y''' + 3y'' - y' = 2x$
14.5	$y^{IV} - y''' = 5(x+2)^2$	14.6	$y^{IV} - 2y''' + y'' = 2x(1-x)$
14.7	$y^{IV} + 2y''' + y'' = x^2 + x - 1$	14.8	$y^V - y^{IV} = 2x + 3$
14.9	$3y^{IV} + y''' = 6x - 1$	14.10	$y^{IV} + 2y''' + y'' = 4x^2$
14.11	$y''' + y'' = 5x^2 - 1$	14.12	$y^{IV} + 4y''' + 4y'' = x - x^2$
14.13	$7y''' - y'' = 12x$	14.14	$y''' + 3y'' + 2y' = 3x^2 + 2x$
14.15	$y''' - y' = 3x^2 - 2x + 1$	14.16	$y''' - y'' = 4x^2 - 3x + 2$
14.17	$y^{IV} - 3y''' + 3y'' - y' = x - 3$	14.18	$y^{IV} + 2y''' + y'' = 12x^2 - 6x$
14.19	$y''' - 4y'' = 32 - 384x^2$	14.20	$y^{IV} + 2y''' + y'' = 2 - 3x^2$
14.21	$y''' + y'' = 49 - 24x^2$	14.22	$y''' - 2y'' = 3x^2 + x - 4$
14.23	$y''' - 13y'' + 12y' = x - 1$	14.24	$y^{IV} + y''' = x$
14.25	$y''' - y'' = 6x + 5$	14.26	$y''' + 3y'' + 2y' = x^2 + 2x + 5$
14.27	$y''' - 5y'' + 6y' = (x-1)^2$	14.28	$y^{IV} - 6y''' + 9y'' = 3x - 1$
14.29	$y''' - 13y'' + 12y' = 18x^2 - 39$	14.30	$y^{IV} + y''' = 12x + 6$

2.3 Solution of an exemplary embodiment

Table 1 - The main types of the 1st order differential equations

Nº	Type of equation	Form of the equation	Solution Method	Remark
1	With separable variables	a) $M_1(x)N_1(y)dx + M_2(x)N_2(y)dy = 0,$ b) $y' = f(x)g(y)$	a) multiply by $\frac{1}{M_2(x)N_1(y)}$ b) $\frac{dy}{g(y)} = f(x)dx$	
2	Linear	$y' + p(x) \cdot y = g(x), (*)$	Bernoulli method. Substitution $y = u \cdot v$, where 1) $u = u(x)$, $v = v(x)$, 2) $y' = u'v + uv'$. $u(x) = e^{-\int p(x)dx}$; Solution $v(x) = \int \frac{g(x)}{u(x)} dx + C$	a) In solving the equation $v' + p(x) \cdot v = 0$, constant C is considered equal to 0.
3	Bernoulli equation	$y' + p(x) \cdot y = g(x) \cdot y^n$, ($n \neq 0; n \neq 1$)	a) Bernoulli method. Substitution $y = u \cdot v$, as for linear equations (*) b) Substitution $z = y^{1-n}$ reduces to linear equation (2) with respect to the new function z .	When $n = 0$ we have a linear equation When $n = 1$ the equation is reduced to an equation with separated variables (1)
4	Equation in total differentials	$P(x, y)dx + Q(x, y)dy = 0$, where $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$	Solution $u(x, y) = C$, where $u(x, y) = \int_{x_0}^x P(x, y_0)dx +$ $+ \int_{y_0}^y Q(x, y)dy$	Check $du = \frac{\partial u}{\partial x} dx +$ $+ \frac{\partial u}{\partial y} dy =$ $= P(x, y)dx +$ $+ Q(x, y)dy$

Task 1. Check whether the function $y = \frac{c^2 - x^2}{2x}$ is the solution of the differential equation $x + y + x \cdot y' = 0$.

Solution: to perform the task, we need a derivative of this function:

$$y' = \left(\frac{c^2 - x^2}{2x} \right)' = \left(\frac{c^2}{2} \cdot \frac{1}{x} - \frac{x}{2} \right)' = -\frac{c^2}{2x^2} - \frac{1}{2}.$$

Substitute y and y' into the equation:

$$x + \frac{c^2 - x^2}{2x} + x \cdot \left(-\frac{c^2}{2x^2} - \frac{1}{2} \right) = 0; \quad x + \frac{c^2}{2x} - \frac{x}{2} - \frac{c^2}{2x} - \frac{x}{2} = 0 \Rightarrow 0 = 0.$$

Answer: the specified function is a solution to the given equation.

Task 2. Find the general solution (general integral) of a differential equation.

$$\frac{dy}{dx} = x^3 y^3.$$

Solution: separate the variables in the equation with separable variables and integrate:

$$\frac{dy}{dx} = x^3 y^3 \Rightarrow dy = x^3 y^3 dx ,$$

$$dy = x^3 y^3 dx / : y^3 \Rightarrow \frac{dy}{y^3} = x^3 dx$$

$$\int \frac{dy}{y^3} = \int x^3 dx; \frac{y^{-2}}{-2} = \frac{x^4}{4} + c; -\frac{1}{2y^2} = \frac{x^4}{4} + c.$$

Thus, $-\frac{1}{2y^2} = \frac{x^4}{4} + c$ is the general integral of the equation.

Task 3. Find the solution of the Cauchy problem $y' = 6y$; $y(0) = 5$ and construct the corresponding integral curve.

Solution: separate the variables in the equation and integrate:

$$y' = 6y \Rightarrow \frac{dy}{dx} = 6y \Rightarrow dy = 6y dx;$$

$$\frac{dy}{y} = 6dx; \quad \int \frac{dy}{y} = \int 6dx; \ln|y| = 6x + c; \quad |y| = e^{6x+c}.$$

We use the initial conditions:

$$|5| = e^{6 \cdot 0 + c}; \quad 5 = e^c; \quad C = \ln 5; \quad y = e^{6x + \ln 5}; \quad y = e^{6x} \cdot e^{\ln 5};$$

$y = 5e^{6x}$ – is a partial solution satisfying initial conditions.

Solution of the Cauchy problem: $y = 5e^{6x}$.

Task 4. Find a solution of the Cauchy problem: $y' = -5y^2$, $y(0) = \frac{1}{5}$.

Solution:

$y' = -5y^2$; $\frac{dy}{dx} = -5y^2$; $dy = -5y^2 dx$ is the equation with separable variables.

Divide by y^2 : $\frac{dy}{y^2} = -5x$; $\int \frac{dy}{y^2} = -5 \cdot \int dx$; $y = \frac{1}{5x + c}$ is the general

solution. Let us find a particular solution satisfying the initial conditions:

$$y(0) = \frac{1}{5}; \quad \frac{1}{5} = \frac{1}{5 \cdot 0 + c}; \quad c = 5.$$

Note. Dividing the equation by y^2 the solution $y=0$ could be lost. By using direct substitution into the initial equation, we make sure that $y=0$ satisfies this equation. In addition, $y=0$ is a special solution, since it is not contained in the general solution.

Answer: $y = \frac{1}{5x + 5}, y = 0$.

Task 5. Find a solution of the Cauchy problem: $y' - 4xy = xe^{x^2}$, $y(0) = 1$:

a) Bernoulli method (substitution $y = u \cdot v$);

b) the method of variation of arbitrary constants; (the student chooses the solution by himself).

Solution: this equation is a linear inhomogeneous equation.

a) in the Bernoulli method we use the substitution $y = u \cdot v$ (where $u = u(x)$, $v = v(x)$ are new unknown functions) $\Rightarrow y' = u'v + uv'$. Substitute in the original equation:

$$u'v + uv' - 4xuv = xe^{x^2} \Rightarrow u'v + u(v' - 4xv) = xe^{x^2}.$$

Choose a function v such that $(v' - 4xv) = 0$. This is the equation with separable variables.

$$\frac{dv}{dx} - 4xv = 0 \Rightarrow \frac{dv}{dx} = 4xv \Rightarrow \frac{dv}{v} = 4xdx \Rightarrow \ln|v| = 2x^2 \Rightarrow v = e^{2x^2}.$$

Given the selected function v , from the original equation we get:

$$\begin{aligned} u'v = xe^{x^2} &\Rightarrow u'e^{2x^2} = xe^{x^2} \Rightarrow \frac{du}{dx} = xe^{-x^2} \Rightarrow \\ &\Rightarrow u = \int xe^{-x^2} dx = \frac{1}{2} \int e^{-x^2} d(x^2) = -\frac{1}{2}e^{-x^2} + C. \end{aligned}$$

Functions $u = u(x)$ and $v = v(x)$ are found.

Consequently, $y = \left(-\frac{1}{2}e^{-x^2} + C\right) \cdot e^{2x^2} = -\frac{1}{2}e^{x^2} + Ce^{2x^2}$ is the general solution.

Using the initial condition $y(0) = 1$, we get: $-\frac{1}{2}e^0 + Ce^0 = 1 \Rightarrow C = \frac{3}{2}$.

The solution of the Cauchy problem: $y = -\frac{1}{2}e^{x^2} + \frac{3}{2}e^{2x^2}$.

b) in the method of variation of arbitrary constants: $y' - 4xy = xe^{x^2}$ is linear inhomogeneous first order equation. Corresponding homogeneous equation: $y' - 4xy = 0$. It is an equation with separable variables.

$$\frac{dy}{dx} - 4xy = 0 \Rightarrow \frac{dy}{dx} = 4xy \Rightarrow \frac{dy}{y} = 4xdx \Rightarrow \int \frac{dy}{y} = \int 4xdx \Rightarrow$$

$\ln|y| = 2x^2 + \ln|C| \Rightarrow y = Ce^{2x^2}$ is a general solution of a linear homogeneous equation.

We are looking for a solution to the inhomogeneous equation in the form: $y = C(x)e^{2x^2}$, where $C(x)$ is an unknown function. Substitute $y = C(x)e^{2x^2}$ and $y' = C'(x)e^{2x^2} + C(x) \cdot 4xe^{2x^2}$ in the original equation:

$$C'(x)e^{2x^2} + C(x) \cdot 4xe^{2x^2} - 4x \cdot C(x)e^{2x^2} = xe^{x^2} \Rightarrow C'(x)e^{2x^2} = xe^{x^2} \Rightarrow$$

$$C'(x) = xe^{-x^2} \Rightarrow C(x) = \int xe^{-x^2} dx = \frac{1}{2} \int e^{-x^2} d(x^2) = -\frac{1}{2}e^{-x^2} + C.$$

So, $y = C(x)e^{2x^2} = \left(-\frac{1}{2}e^{-x^2} + C\right) \cdot e^{2x^2} = -\frac{1}{2}e^{x^2} + Ce^{2x^2}$ is the general solution

of the original equation. Using the initial condition $y(0) = 1$, we get:

$$-\frac{1}{2}e^0 + Ce^0 = 1 \Rightarrow C = \frac{3}{2} \text{ or } y = -\frac{1}{2}e^{x^2} + \frac{3}{2}e^{2x^2}.$$

Task 6. Find the general solution of the Bernoulli equation $y' + y = xy^{\frac{1}{2}}$.

Solution: $y' + y = xy^{\frac{1}{2}}$ is the Bernoulli equation, where $n=1/2$. Perform a substitution $z = y^{1-n} = y^{1-\frac{1}{2}} = y^{\frac{1}{2}}$. $y = z^2$; $y' = 2z \cdot z'$ (see Table 2). Substitute this into the original equation:

$$2z \cdot z' + z^2 = x \cdot z; z' + \frac{1}{2} \cdot z = \frac{1}{2} \cdot x \text{ is a linear equation.}$$

Substitution: $z = uv$; $z' = u'v + uv'$;

$$u'v + uv' + \frac{1}{2}uv = \frac{1}{2}x; v' + \frac{1}{2}v = 0; \frac{dv}{v} + \frac{dx}{2} = 0; \ln v + \frac{x}{2} = 0; v = e^{-\frac{x}{2}};$$

$$u' \cdot e^{-\frac{x}{2}} = \frac{x}{2}; \Rightarrow \frac{du}{dx} \cdot e^{-\frac{x}{2}} = \frac{x}{2};$$

$$du = \frac{x}{2} \cdot e^{\frac{x}{2}} dx; \quad u = \int \frac{x}{2} \cdot e^{\frac{x}{2}} dx = xe^{\frac{x}{2}} - 2e^{\frac{x}{2}} + c.$$

$$\text{So, } z = u \cdot v = e^{-\frac{x}{2}} \cdot \left(xe^{\frac{x}{2}} - 2e^{\frac{x}{2}} + c \right) = x + ce^{-\frac{x}{2}} - 2.$$

The solution of the Cauchy problem: $y = z^2$, i.e. $y = \left(x + ce^{-\frac{x}{2}} - 2 \right)^2$.

Task 7. Find the general integral of the differential equation in total differentials

$$(x + y - 1)dx + (e^y + x)dy = 0.$$

Solution:

$$P(x, y) = x + y + 1, \quad Q(x, y) = e^y + x, \quad \frac{\partial Q}{\partial x} = 1, \quad \frac{\partial P}{\partial y} = 1 \Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y},$$

therefore, the total differential condition for function u is satisfied:

$$du = (x + y - 1)dx + (e^y + 1)dy.$$

We find an unknown function u by the formula:

$$u = \int_{x_0}^x P(x; y)dx + \int_{y_0}^y Q(x_0; y)dy.$$

We take $x_0 = 0, y_0 = 0$:

$$u = \int_0^x (x - y - 1)dx + \int_0^y e^y dy = \left(\frac{x^2}{2} + xy - x \right) \Big|_0^x + e^y \Big|_0^y = \frac{x^2}{2} + xy - x + e^y - 1$$

As $du = 0$, then $u = C$ is the general integral of the differential equation.

$$\text{General solution: } \frac{x^2}{2} + xy - x + e^y - 1 = C.$$

Table 2 - Higher order differential equations, allowing lowering the order

Nº	Formula of the equation	Explanations	Estimated Replacement (manual)
1	$y^{(n)} = f(x)$	The derivative of an unknown function is explicitly expressed through a function depending on x.	Integrate n times
2	$F(y, y', y'') = 0$	There is no independent variable in the equation. x .	$y'(x) = u(y),$ $y''(x) = u(y)u'(y)$
3	$F(x, y', y'') = 0$	There is no unknown function $y(x)$ in the equation.	$y'(x) = u(x),$ $y''(x) = u'(x)$
4	$F(x, y^{(k)}, y^{(k+1)}, \dots, y^{(n+k)}) = 0$	The equation has no unknown function and derivatives up to the $(k-1)$ order.	$y^{(k)}(x) = u(x),$ $y^{(k+1)}(x) = u'(x), \dots,$ $y^{(n+k)}(x) = u^{(n)}(x)$

Task 8. Find the general solution of a differential equation $y'' = \sin 2x \cos 2x$.

Solution:

$y'' = \sin 2x \cos 2x$ is the second-order equation of the form $y'' = f(x)$. We lower the order by double integration:

$$y' = \int \sin 2x \cos 2x dx = \frac{1}{2} \int \sin 4x dx = -\frac{1}{8} \cos 4x + C_1;$$

$$y' = \int \left(-\frac{1}{8} \cos 4x + C_1 \right) dx = -\frac{1}{8} \int \cos 4x dx + \int C_1 dx = -\frac{1}{32} \sin 4x + C_1 x + C_2.$$

Answer: $y = -\frac{1}{32} \sin 4x + C_1 x + C_2$ is the general solution.

Task 9. Find the general solution of a differential equation $(1+x^2)y'' - 2xy' = 0$.

Solution:

$(1+x^2)y'' - 2xy' = 0$ is the second order equation with explicitly missing function y (Table 2). In this case, we apply the replacement: $y' = p$. Then

$$y'' = \frac{dp}{dx} = p'.$$

The equation has the form: $(1+x^2)p' - 2xp = 0$, $(1+x^2)dp - 2xpdx = 0$ is the equation with separable variables. Let's divide both parts by $p(1+x^2)$, we get:

$$\frac{dp}{p} = \frac{2xdx}{1+x^2}, \quad \frac{dp}{p} = \frac{d(1+x^2)}{1+x^2}, \quad \ln|p| = \ln(1+x^2) + \ln|C_1|, \quad p = C_1(1+x^2).$$

Since $p = \frac{dy}{dx}$, then $\frac{dy}{dx} = C_1(1+x^2)$ is the equation with separable variables. $dy = C_1(1+x^2)dx$, $y = \int C_1(1+x^2)dx$, $y = C_1\left(x + \frac{x^3}{3}\right) + C_2$ is the general solution.

Note - in the process of dividing by $p(1+x^2)$ we could lose the solutions $p=0$ and $1+x^2=0$. The first gives: $y'=0 \Rightarrow y=C$, but this solution is in the general solution when $C_1=0$. The second equality $1+x^2=0$ is impossible for real x .

Task 10. Find the general solution of a homogeneous linear differential equation with constant coefficients.

- a) $y'' - 11y' + 10y = 0$;
- b) solve the Cauchy problem: $9y'' + 6y' + y = 0$, $y(0) = 1$, $y'(0) = 2$;
- c) $y'' + 2y' + 10y = 0$.

Linear homogeneous differential equation of the second order with constant coefficients has the form: $y'' + py' + qy = 0$.

Its characteristic equation: $k^2 + pk + q = 0$, where $D = p^2 - 4q$ is its discriminant. Then the structure of the general solution of a linear homogeneous differential equation with constant coefficients depends on the type of roots of the corresponding characteristic equation (table 3).

Table 3 - The structure of the general solution of a linear homogeneous differential equation with constant coefficients

D	Roots of the characteristic equation $k^2 + pk + q = 0$	Fundamental system of solutions	The general solution of a homogeneous linear high order differential equation with constant coefficients $y_{g.h.} = C_1y_1 + C_2y_2$
$D > 0$	$k_1 = \frac{-p - \sqrt{D}}{2}$, $k_2 = \frac{-p + \sqrt{D}}{2}$	$y_1 = e^{k_1 x}$, $y_2 = e^{k_2 x}$	$y_{g.h.} = C_1 e^{k_1 x} + C_2 e^{k_2 x}$

$D=0$	$k_1 = k_2 = \mu$ $\mu = \frac{-p}{2}$	$y_1 = e^{\mu x},$ $y_2 = xe^{\mu x}$	$y_{g.h.} = C_1 e^{\mu x} + C_2 x e^{\mu x}$
$D<0$	$k_{1,2} = \alpha \pm i\beta$ $\alpha = \frac{-p}{2}, \beta = \frac{\sqrt{ D }}{2}$	$y_1 = e^{\alpha x} \cos \beta x,$ $y_2 = e^{\alpha x} \sin \beta x$	$y_{g.h.} = C_1 e^{\alpha x} \cos \beta x$ $+ C_2 e^{\alpha x} \sin \beta x$

Solution:

a) $y'' - 11y' + 10y = 0 \rightarrow k^2 - 11k + 10 = 0$ is the characteristic equation,

$$\begin{cases} k_1 = 1 \\ k_2 = 10 \end{cases} \text{ -- different real roots of the characteristic equation.}$$

Therefore, the general solution is $y_{g.h.} = C_1 e^{1x} + C_2 e^{10x}$.

b) $9y'' + 6y' + y = 0, y(0) = 1, y'(0) = 2 \rightarrow$

the characteristic equation $9k^2 + 6k + 1 = 0$ has equal roots $k_1 = k_2 = -\frac{1}{3}$, so the general solution is $y_{g.h.} = C_1 e^{-\frac{1}{3}x} + C_2 x e^{-\frac{1}{3}x}$.

To use the second initial condition we differentiate the general solution:

$$y' = -\frac{1}{3} C_1 e^{-\frac{1}{3}x} + C_2 e^{-\frac{1}{3}x} - \frac{1}{3} C_2 x e^{-\frac{1}{3}x}.$$

Next, we use the initial conditions:

$$\left. \begin{array}{l} 1 = C_1 e^0 + C_1 \cdot 0 \cdot e^0, \\ 2 = -\frac{1}{3} C_1 e^0 + C_2 e^0 - \frac{1}{3} C_1 \cdot 0 \cdot e^0 \end{array} \right\} \Rightarrow C_1 = 1; \quad C_2 = \frac{7}{3}.$$

A particular solution of the Cauchy problem has the form:

$$y_{p.s.} = e^{-\frac{1}{3}x} + \frac{7}{3} x e^{-\frac{1}{3}x}.$$

c) for differential equation $y'' + 2y' + 10y = 0$ the characteristic equation $k^2 + 2k + 10 = 0$ has complex roots in the form: $k_{1,2} = -1 \pm 3i \Rightarrow \alpha = -1, \beta = 3$.

Consequently, $y_{g.h.} = e^{-x} (C_1 \cos 3x + C_2 \sin 3x)$ is the general solution.

Task 11. Find the general solution of the differential equation $y'' - 2y' + 2y = 6e^{2x}$.

Solution: this is a second-order linear inhomogeneous differential equation with constant coefficients: $y'' + py' + qy = f(x)$, where the right-hand side has the form:

$$f(x) = e^{ax} [P_m(x) \cos bx + Q_n(x) \sin bx].$$

The structure of the general solution of this equation:

$$y = y_{g.h.} + y_{p.i.},$$

consists of $y_{g.h.}$ – the general solution of the corresponding homogeneous equation and the particular solution

$$y_{p.i.} = x^r e^{ax} (\tilde{P}_k(x) \cos bx + \tilde{Q}_k(x) \sin bx),$$

where r is the number of numbers $a + bi$ among the roots of the characteristic equation, $k = \max\{m, n\}$.

Algorithm for finding a general solution.

First we find $y_{g.h.}: k^2 - 2k + 2 = 0$ – characteristic equation of the corresponding homogeneous equation, $k_{1,2} = 1 \pm i$ – the roots of this equation, consequently, $y_{g.h.} = e^x (C_1 \cos x + C_2 \sin x)$.

Next, we find $y_{p.i.}$ using the method of indefinite coefficients by the form of the right-hand side $f(x)$ of the inhomogeneous equation.

Since $f(x) = ae^{mx} = 6e^{2x}$ ($a = 6$, $m = 2$) and $m = 2$ is not the root of the characteristic equation, then, in accordance to theory, we will look for a particular solution in the form $y_{p.i.} = Ae^{2x}$. To find the unknown coefficient A , we substitute

$$y_{p.i.} = Ae^{2x}, \quad y'_{p.i.} = 2Ae^{2x}, \quad y''_{p.i.} = 4Ae^{2x}$$

into the original equation:

$$4Ae^{2x} - 2 \cdot 2Ae^{2x} + 2Ae^{2x} = 6e^{2x}, \quad 2Ae^{2x} = 6e^{2x},$$

$$2A = 6, \quad A = 3.$$

Consequently,

$$y_{p.i.} = 3e^{2x}, \quad y = e^x (C_1 \cos x + C_2 \sin x) + 3e^{2x} \quad \text{is the general solution.}$$

$$\text{Answer: } y = e^x (C_1 \cos x + C_2 \sin x) + 3e^{2x}.$$

Task 12. Solve a system of differential equations by the elimination method:

$$\begin{cases} y'_1 = -9y_1 + 4y_2 \\ y'_2 = 9y_1 - 4y_2 \end{cases}.$$

Solution: let's differentiate with respect to x the first equation from the system: $y''_1 = -9y'_1 + 4y'_2$. To eliminate y'_2 let's summarize the lines in the given system: $y'_1 + y'_2 = 0 \Rightarrow y'_2 = -y'_1$.

Consequently, $y''_1 = -9y'_1 + 4(-y'_1)$; $y''_1 + 13y'_1 = 0$ is linear homogeneous differential equation with constant coefficients.

$$k^2 + 13k = 0 \Rightarrow k_1 = 0; k_2 = -13.$$

$$\text{Then } y_1 = C_1 e^{0x} + C_2 e^{-13x} = C_1 + C_2 e^{-13x}.$$

As $y'_1 = -13C_2 e^{-13x}$, then we find from the first equation y_2 :

$$4y_2 = y'_1 + 9y_1 \Rightarrow y_2 = \frac{1}{4}(-13C_2 e^{-13x} + 9(C_1 + C_2 e^{-13x})) = \frac{9}{4}C_1 - C_2 e^{-13x}.$$

$$\text{Answer: } \begin{cases} y_1 = C_1 + C_2 e^{-13x} \\ y_2 = \frac{9}{4}C_1 - C_2 e^{-13x}. \end{cases}$$

3 Calculation-graphical work №3. Series

Purpose: master the fundamental concepts and methods of the theory of series. Get skills to apply signs of convergence in the study of series.

3.1 Theoretical questions

- 1 Numerical series. Convergence and sum of a series. A necessary condition for the convergence of the series.
- 2 Comparison test.
- 3 D'Alembert's test. The radical Cauchy criterion. The integral Cauchy test.
- 4 Alternating series. Leibniz theorem. Estimate of the remaining tail of the series.
- 5 Alternating series. Absolutely and conditionally convergent series.
- 6 Functional series. Domain of convergence.
- 7 Power series. Abel's theorem. Interval and radius of convergence of power series.
- 8 Taylor Series. Expansion in powers of x the following functions e^x , $\sin x$, $\cos x$, $\ln(1+x)$, $(1+x)^m$.

9 Fourier series. Fourier coefficients.

3.2 Calculated tasks

Task 1. For series $\sum_{n=1}^{\infty} u_n$:

- a) make a general term formula u_n and write the first five terms;
- b) write the n -th partial sum S_n of the series;
- c) write the remaining tail r_n of the series;
- d) check the necessary condition for convergence of the series.

1.1	$\sum_{n=1}^{\infty} \frac{1}{(n+1) \cdot 3^n}$	1.2	$\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$	1.3	$\sum_{n=1}^{\infty} \frac{1}{(2n+1) \cdot 5^n}$
1.4	$\sum_{n=1}^{\infty} \frac{2n+1}{7^n}$	1.5	$\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$	1.6	$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}$
1.7	$\sum_{n=1}^{\infty} \frac{3}{\ln(n+1)}$	1.8	$\sum_{n=1}^{\infty} \frac{2^n}{5^n}$	1.9	$\sum_{n=1}^{\infty} \frac{2^n+4^n}{10^n}$
1.10	$\sum_{n=1}^{\infty} \frac{1}{(3n+1)^2}$	1.11	$\sum_{n=1}^{\infty} \frac{1}{3n^3+1}$	1.12	$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n+5}}$
1.13	$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$	1.14	$\sum_{n=1}^{\infty} \frac{1}{(2n+1) \cdot (3n+3)}$	1.15	$\sum_{n=1}^{\infty} \frac{n}{n+1}$
1.16	$\sum_{n=1}^{\infty} \frac{2n}{n+5}$	1.17	$\sum_{n=1}^{\infty} \frac{n}{(3n+1)}$	1.18	$\sum_{n=1}^{\infty} \frac{5^n}{3^n}$
1.19	$\sum_{n=1}^{\infty} \frac{6}{(5n+1)}$	1.20	$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3^n}}$	1.21	$\sum_{n=1}^{\infty} \frac{1}{\sqrt{(3n+1)}}$
1.22	$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{(n+2)}}$	1.23	$\sum_{n=1}^{\infty} \frac{1}{(5n+1)^5}$	1.24	$\sum_{n=1}^{\infty} \frac{1}{(2n-3) \cdot 6^n}$
1.25	$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2]{(2n+1)} \cdot 3^n}$	1.26	$\sum_{n=1}^{\infty} \frac{1}{\ln(2n+1)}$	1.27	$\sum_{n=1}^{\infty} \frac{1}{\ln^2(n+1)}$
1.28	$\sum_{n=1}^{\infty} \frac{1}{(n^2+2n-1)}$	1.29	$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}$	1.30	$\sum_{n=1}^{\infty} \frac{3}{(2n+1)^2}$

Task 2. Compare with Dirichlet series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, i.e. find the parameter p , and investigate the convergence of the series.

2.1	$\sum_{n=1}^{\infty} \frac{1}{(n^2 + 2n - 1)}$	2.2	$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}$	2.3	$\sum_{n=1}^{\infty} \frac{3}{(2n+1)^2}$
2.4	$\sum_{n=1}^{\infty} \frac{6}{(5n+1)}$	2.5	$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3+n^3}}$	2.6	$\sum_{n=1}^{\infty} \frac{1}{\sqrt{(3n^3+1)}}$
2.7	$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{(n+2)}}$	2.8	$\sum_{n=1}^{\infty} \frac{1}{(5n+1)^5}$	2.9	$\sum_{n=1}^{\infty} \frac{1}{(2n-3)\cdot(6n-2)}$
2.10	$\sum_{n=1}^{\infty} \frac{1}{(3n+1)^2}$	2.11	$\sum_{n=1}^{\infty} \frac{1}{3n^3+1}$	2.12	$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(3n+3)}$
2.13	$\sum_{n=2}^{\infty} \frac{1}{\sqrt{(n-1)(n+2)}}$	2.14	$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n+5}}$	2.15	$\sum_{n=1}^{\infty} \left(\frac{n}{2n^2+1}\right)^2$
2.16	$\sum_{n=1}^{\infty} \frac{n^3}{(3n+1)^2}$	2.17	$\sum_{n=1}^{\infty} \sqrt[5]{\frac{1}{3n^3+1}}$	2.18	$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+5}}$
2.19	$\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n-1}(n^2+2)}$	2.20	$\sum_{n=1}^{\infty} \frac{2n+1}{(3n+3)}$	2.21	$\sum_{n=1}^{\infty} \frac{n^5}{n+1}$
2.22	$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(3n+1)^2}$	2.23	$\sum_{n=1}^{\infty} \sqrt[4]{\frac{n}{3n^3+1}}$	2.24	$\sum_{n=1}^{\infty} \frac{n}{(2n+1)^2}$
2.25	$\sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n^2+2)}$	2.26	$\sum_{n=1}^{\infty} \frac{2n^2+1}{(6n+3)}$	2.27	$\sum_{n=1}^{\infty} \frac{n^5}{(n+1)^3}$
2.28	$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$	2.29	$\sum_{n=1}^{\infty} \left(\frac{n}{3n^3+1}\right)^4$	2.30	$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(2n-1)^3}$

Task 3. Investigate for convergence using the D'Alembert's test.

3.1	$\sum_{n=1}^{\infty} \frac{1}{(n+1)\cdot 3^n}$	3.2	$\sum_{n=2}^{\infty} \frac{2^n}{(n-1)!}$	3.3	$\sum_{n=1}^{\infty} \frac{1}{(2n+1)\cdot 5^n}$
3.4	$\sum_{n=1}^{\infty} \frac{2n+1}{7^n}$	3.5	$\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$	3.6	$\sum_{n=1}^{\infty} \frac{n!}{(2n-1)^3}$
3.7	$\sum_{n=1}^{\infty} \frac{3n}{(n+7)!}$	3.8	$\sum_{n=1}^{\infty} \frac{n!}{5^n}$	3.9	$\sum_{n=1}^{\infty} \frac{(n-1)^2}{10^n}$
3.10	$\sum_{n=1}^{\infty} \frac{n}{(n+1)\cdot 3^n}$	3.11	$\sum_{n=1}^{\infty} \frac{\sqrt{2n+1}}{(n+2)!}$	3.12	$\sum_{n=1}^{\infty} \frac{(n+10)^3}{5^n}$

3.13	$\sum_{n=1}^{\infty} \frac{2n+1}{n!}$	3.14	$\sum_{n=2}^{\infty} \frac{4^{n-1}}{(n-1)!}$	3.15	$\sum_{n=1}^{\infty} \frac{2n!}{\sqrt{(2n-1)}}$
3.16	$\sum_{n=1}^{\infty} \frac{3n}{\sqrt{2^n}}$	3.17	$\sum_{n=2}^{\infty} \frac{(n+3)!}{3^{n-1}}$	3.18	$\sum_{n=1}^{\infty} \frac{(2n-1)^2}{8^n}$
3.19	$\sum_{n=1}^{\infty} \frac{3^n}{(n+1)}$	3.20	$\sum_{n=2}^{\infty} \frac{5^{n-1}}{(n-1)!}$	3.21	$\sum_{n=1}^{\infty} \frac{1}{(6n+1) \cdot 7^n}$
3.22	$\sum_{n=1}^{\infty} \frac{7n+7}{2^n}$	3.23	$\sum_{n=1}^{\infty} \frac{6^n}{(n+1)!}$	3.24	$\sum_{n=1}^{\infty} \frac{n!}{(2n-1)(n+1)}$
3.25	$\sum_{n=3}^{\infty} \frac{n-1}{(n-2)!}$	3.26	$\sum_{n=1}^{\infty} \frac{n^2 - 5}{5^{n/2}}$	3.27	$\sum_{n=1}^{\infty} \frac{(2n-1)^2}{10^n}$
3.28	$\sum_{n=1}^{\infty} \frac{3^n n}{(n^2 + 7)}$	3.29	$\sum_{n=2}^{\infty} \frac{n!}{5^n (n-1)}$	3.30	$\sum_{n=1}^{\infty} \frac{\sqrt[3]{(n-1)}}{10^n}$

Task 4. Investigate for convergence using the radical Cauchy criterion.

4.1	$\sum_{n=1}^{\infty} \frac{1}{3^n}$	4.2	$\sum_{n=1}^{\infty} \left(\operatorname{tg} \frac{\pi}{(2n-1)} \right)^{2n}$	4.3	$\sum_{n=1}^{\infty} \frac{(2n+1)^n}{7^n}$
4.4	$\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+3)} \right)^{3n}$	4.5	$\sum_{n=1}^{\infty} \left(\operatorname{arctg} \frac{1}{(n+1)} \right)^n$	4.6	$\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+3)} \right)^n$
4.7	$\sum_{n=1}^{\infty} \frac{3}{(\ln(n+2))^n}$	4.8	$\sum_{n=1}^{\infty} \left(\frac{3n^2 - n - 1}{7n^2 + 3n + 4} \right)^n$	4.9	$\sum_{n=1}^{\infty} \left(\sin \frac{\pi}{n^3} \right)^{2n}$
4.10	$\sum_{n=1}^{\infty} \left(\frac{n+1}{4n} \right)^{3n}$	4.11	$\sum_{n=1}^{\infty} \left(\frac{n-2}{2n+1} \right)^n \cdot \frac{1}{5^n}$	4.12	$\sum_{n=1}^{\infty} \left(\frac{3n-2}{2n+1} \right)^{2n}$
4.13	$\sum_{n=1}^{\infty} \frac{7^n}{(8n+14)^n}$	4.14	$\sum_{n=1}^{\infty} \left(\arcsin \frac{1}{3^n} \right)^n$	4.15	$\sum_{n=1}^{\infty} \left(\operatorname{tg} \frac{\pi}{5^n} \right)^{3n}$
4.16	$\sum_{n=1}^{\infty} \frac{3^n}{(n+2)^{3n}}$	4.17	$\sum_{n=1}^{\infty} \left(\arcsin \left(\frac{n+1}{4n+1} \right) \right)^n$	4.18	$\sum_{n=1}^{\infty} \left(\sin \frac{\pi}{\sqrt{n}} \right)^{2n}$
4.19	$\sum_{n=1}^{\infty} \frac{10^n}{(\ln(n+5))^2}$	4.20	$\sum_{n=1}^{\infty} \left(\frac{n^2 + 5n - 1}{7n^2 + 3n + 4} \right)^n$	4.21	$\sum_{n=1}^{\infty} \left(\frac{n-2}{2n+1} \right)^{2n}$
4.22	$\sum_{n=1}^{\infty} \frac{(n+2)^n 7^n}{(n+14)^n}$	4.23	$\sum_{n=1}^{\infty} \left(\operatorname{arctg} \frac{\pi}{2n-4} \right)^{3n}$	4.24	$\sum_{n=1}^{\infty} \left(\arcsin \frac{1}{4^n} \right)^n$
4.25	$\sum_{n=1}^{\infty} \frac{5^n}{(n+5)^{3n}}$	4.26	$\sum_{n=1}^{\infty} \left(\arcsin \left(\frac{5n+1}{4n+1} \right) \right)^n$	4.27	$\sum_{n=1}^{\infty} \left(\sin \frac{\pi}{5n+1} \right)^n$

4.28	$\sum_{n=1}^{\infty} \left(\frac{3n-2}{n+1} \right)^{n^2}$	4.29	$\sum_{n=1}^{\infty} \left(\frac{4}{\ln(2n+3)} \right)^{3n}$	4.30	$\sum_{n=1}^{\infty} \left(\frac{3n^2-n-1}{n^2+3n-5} \right)^n$
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Task 5. Investigate for convergence using the integral Cauchy test.

5.1	$\sum_{n=1}^{\infty} \frac{1}{(3n+1)\ln(3n+1)}$	5.2	$\sum_{n=1}^{\infty} \frac{1}{(n+2)\ln^2(n+2)}$
5.3	$\sum_{n=1}^{\infty} \frac{1}{(5n+1)\sqrt{\ln(5n+1)}}$	5.4	$\sum_{n=1}^{\infty} \frac{1}{(5n+1)\ln^2(5n+1)}$
5.5	$\sum_{n=1}^{\infty} \frac{1}{(n+2)\ln^4(n+2)}$	5.6	$\sum_{n=1}^{\infty} \frac{1}{(4n+1)\sqrt[3]{\ln(4n+1)}}$
5.7	$\sum_{n=1}^{\infty} \frac{1}{(3n+1)\sqrt{\ln(3n+1)}}$	5.8	$\sum_{n=1}^{\infty} \frac{1}{(n+12)\ln^5(n+12)}$
5.9	$\sum_{n=1}^{\infty} \frac{1}{(5n+1)\sqrt{\ln^3(5n+1)}}$	5.10	$\sum_{n=1}^{\infty} \frac{1}{(3n+5)\sqrt[3]{\ln(3n+5)}}$
5.11	$\sum_{n=1}^{\infty} \frac{1}{(n+12)\sqrt[3]{\ln^2(n+12)}}$	5.12	$\sum_{n=1}^{\infty} \frac{1}{(5n+5)\sqrt{\ln^2(5n+5)}}$
5.13	$\sum_{n=1}^{\infty} \frac{1}{(5n+1)\sqrt[3]{\ln^2(5n+1)}}$	5.14	$\sum_{n=1}^{\infty} \frac{1}{(n+3)\sqrt{\ln^4(n+3)}}$
5.15	$\sum_{n=1}^{\infty} \frac{1}{n \cdot \sqrt[3]{\ln^4(4n)}}$	5.16	$\sum_{n=1}^{\infty} \frac{1}{(3n+2)\sqrt{\ln^5(3n+2)}}$
5.17	$\sum_{n=1}^{\infty} \frac{1}{(2n+12)\ln^3(2n+12)}$	5.18	$\sum_{n=1}^{\infty} \frac{1}{n \cdot \sqrt{\ln^3(5n)}}$
5.19	$\sum_{n=1}^{\infty} \frac{1}{(2n+1)\ln^4(2n+1)}$	5.20	$\sum_{n=1}^{\infty} \frac{1}{(n+2)\sqrt[4]{\ln(n+2)}}$
5.21	$\sum_{n=1}^{\infty} \frac{1}{(7n+1)\sqrt{\ln^7(7n+1)}}$	5.22	$\sum_{n=2}^{\infty} \frac{1}{n \ln^{12} n}$
5.23	$\sum_{n=1}^{\infty} \frac{1}{(n+6)\ln^8(n+6)}$	5.24	$\sum_{n=1}^{\infty} \frac{1}{(n+4)\sqrt[3]{\ln^4(n+4)}}$
5.25	$\sum_{n=3}^{\infty} \frac{1}{(2n-3)\ln(2n-3)}$	5.26	$\sum_{n=1}^{\infty} \frac{1}{(n+11)\sqrt[3]{\ln^5(n+11)}}$

5.27	$\sum_{n=1}^{\infty} \frac{1}{(5n+1)\sqrt{\ln^3(5n+1)}}$	5.28	$\sum_{n=1}^{\infty} \frac{1}{(n+5)\sqrt[6]{\ln(n+5)}}$
5.29	$\sum_{n=1}^{\infty} \frac{1}{(n+12)\sqrt[3]{\ln^6(n+12)}}$	5.30	$\sum_{n=1}^{\infty} \frac{1}{(5n-3)\sqrt{\ln^5(5n-3)}}$

Task 6. Investigate alternating series for conditional or absolute convergence.

6.1	$\sum_{n=1}^{\infty} \frac{(-1)^n}{(3n+1)^2}$	6.2	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n^3 + 1}$	6.3	$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+5}}$
6.4	$\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)(n+2)}$	6.5	$\sum_{n=1}^{\infty} \frac{(-1)^n n}{(2n+1) \cdot (3n+3)}$	6.6	$\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n+1}$
6.7	$\sum_{n=1}^{\infty} \frac{(-1)^n 2n}{n^2 + 5}$	6.8	$\sum_{n=1}^{\infty} \frac{(-1)^n n}{(3n+1)}$	6.9	$\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{3^n}$
6.10	$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{5n+1}}$	6.11	$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{3^n}}$	6.12	$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{(3n+1)}}$
6.13	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{(n+2)}}$	6.14	$\sum_{n=1}^{\infty} \frac{(-1)^n}{(5n+1)^5}$	6.15	$\sum_{n=1}^{\infty} \frac{(-1)^n (n+2)}{6^n}$
6.16	$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)}$	6.17	$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$	6.18	$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{5^n}$
6.19	$\sum_{n=1}^{\infty} \frac{(-1)^n}{7n\sqrt{n+5}}$	6.20	$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(n+1)!}$	6.21	$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3}$
6.22	$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[2]{(2n+1)}}$	6.23	$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(2n+1)}$	6.24	$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[5]{n}}$
6.25	$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$	6.26	$\sum_{n=1}^{\infty} \frac{(-2)^n}{5^n}$	6.27	$\sum_{n=1}^{\infty} \frac{(-2)^n}{10^n}$
6.28	$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2 + 2n - 1)}$	6.29	$\sum_{n=1}^{\infty} \frac{(-1)^n 6^n}{(n-1)!}$	6.30	$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{(2n+1)^2}}$

Task 7. Given power series $\sum_{n=1}^{\infty} a_n (x - x_0)^n$. Find the radius and interval of convergence of the series.

7.1	$\sum_{n=1}^{\infty} \frac{(x+1)^n}{(\ln(n+2))^n}$	7.2	$\sum_{n=1}^{\infty} \left(\frac{n+1}{4n}\right)^{3n} (x-3)^n$
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7.3	$\sum_{n=1}^{\infty} \left(\sin \frac{\pi}{n^3} \right)^{2n} (x+1)^n$	7.4	$\sum_{n=1}^{\infty} \left(\frac{3n^2}{7n^2 + 3n + 4} \right)^n (x-2)^n$
7.5	$\sum_{n=1}^{\infty} \left(\frac{n}{(n+3)} \right)^{3n} \cdot (2x-1)^n$	7.6	$\sum_{n=1}^{\infty} \left(\frac{3n-2}{2n+1} \right)^{2n} (2x+1)^n$
7.7	$\sum_{n=1}^{\infty} \frac{7^n (x-3)^n}{(8n+14)}$	7.8	$\sum_{n=1}^{\infty} \left(\arcsin \frac{1}{3^n} \right) x^n$
7.9	$\sum_{n=1}^{\infty} \left(\operatorname{tg} \frac{\pi}{5^n} \right)^{3n} \cdot (x-1)^n$	7.10	$\sum_{n=1}^{\infty} \frac{n}{(n+1) \cdot 3^n} (x-1)^n$
7.11	$\sum_{n=1}^{\infty} \frac{\sqrt{2n+1}}{n!} (5x+1)^n$	7.12	$\sum_{n=1}^{\infty} \frac{(n+10)^3}{5^n} (3x-1)^n$
7.13	$\sum_{n=1}^{\infty} \frac{2n+1}{n!} x^n$	7.14	$\sum_{n=2}^{\infty} \frac{4^{n-1} (x+1)^n}{\sqrt[3]{n-1}}$
7.15	$\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{(2n-1)}} (x+2)^n$	7.16	$\sum_{n=1}^{\infty} \frac{3n}{2^n} x^{2n}$
7.17	$\sum_{n=2}^{\infty} \frac{(n+3)! x^n}{3^{n-1}}$	7.18	$\sum_{n=1}^{\infty} \sqrt[5]{\frac{1}{3n^3+1}} (4x+1)^n$
7.19	$\sum_{n=1}^{\infty} \frac{3^n x^{2n}}{(n+1)}$	7.20	$\sum_{n=1}^{\infty} \frac{2n+1}{(3n+3)} (x-5)^n$
7.21	$\sum_{n=1}^{\infty} \frac{(x-5)^n}{(6n+1) \cdot 7^n}$	7.22	$\sum_{n=1}^{\infty} \frac{7n+7}{2^n} (2x+1)^n$
7.23	$\sum_{n=1}^{\infty} \frac{5^n}{(3n+1)^2} (x+3)^n$	7.24	$\sum_{n=1}^{\infty} \sqrt[5]{\frac{1}{3n^3+1}} (2x-3)^n$
7.25	$\sum_{n=1}^{\infty} \frac{(n-1)x^{2n}}{2^n (n^2+2)}$	7.26	$\sum_{n=1}^{\infty} \frac{2n+1}{(3n+3)} x^n$
7.27	$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3^n} x^n$	7.28	$\sum_{n=1}^{\infty} \left(\frac{n}{3n^3+1} \right)^n (5-2x)^n$
7.29	$\sum_{n=1}^{\infty} \operatorname{tg} \frac{\pi}{6^n} (x+1)^n$	7.30	$\sum_{n=1}^{\infty} \frac{3^n (2x-3)^n}{(9n+14)^n}$

Task 8. Calculate the sum of a series with precision $\alpha=0,001$.

8.1	$\sum_{n=0}^{\infty} \left(-\frac{2}{5} \right)^n, \alpha = 0,01$	8.2	$\sum_{n=0}^{\infty} \frac{\cos \pi \cdot n}{(n^3+1)^2}, \alpha = 0,001$
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8.3	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2^n \cdot n!},$	8.4	$\sum_{n=0}^{\infty} \frac{(-1)^n}{1+n^3}, \alpha = 0,01$
8.5	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n)! \cdot 2^n}, \alpha = 0,00001$	8.6	$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^3 \cdot (n+3)}, \alpha = 0,001$
8.7	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{3n!}, \alpha = 0,01$	8.8	$\sum_{n=1}^{\infty} (-1)^n \frac{\sin 3^n}{3^n}, \alpha = 0,001$
8.9	$\sum_{n=0}^{\infty} (-1)^n \frac{1}{1+n^3}, \alpha = 0,01$	8.10	$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)^n},$
8.11	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n)!}, \alpha = 0,001$	8.12	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{4^{n+1} \cdot 2^n}, \alpha = 0,001$
8.13	$\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n, \alpha = 0,1$	8.14	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{4^n \cdot n!},$
8.15	$\sum_{n=0}^{\infty} \frac{\cos \pi \cdot n}{(n^3 + 1)^2}, \alpha = 0,001$	8.16	$\sum_{n=1}^{\infty} (-1)^n \frac{1}{4^n (2n+1)}, \alpha = 0,001$
8.17	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3n^2}, \alpha = 0,01$	8.17	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^n \cdot n!}, \alpha = 0,001$
8.19	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n \cdot 2^n}, \alpha = 0,01$	8.20	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2 (n+3)}, \alpha = 0,01$
8.21	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n+1)!},$	8.22	$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)^n}, \alpha = 0,001$
8.23	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n!(2n+1)}, \alpha = 0,001$	8.24	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n^3 (n+1)}, \alpha = 0,01$
8.25	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n)^3}, \alpha = 0,001$	8.26	$\sum_{n=2}^{\infty} \frac{\cos \pi \cdot n}{3^n \cdot n}, \alpha = 0,001$
8.27	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!}, \alpha = 0,01$	8.28	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n)!} \cdot \frac{1}{n!}$
8.29	$\sum_{n=2}^{\infty} \frac{\sin \left[\frac{\pi}{2} (-1^n) \right]}{2^n \cdot n}, \alpha = 0,01$	8.30	$\sum_{n=2}^{\infty} \frac{\sin \left(\frac{\pi}{2} + \pi \cdot n \right)}{n^3}, \alpha = 0,01$

Task 9. Expand in Fourier series periodic function $f(x)$ with period $w=2l$, given in the specified interval.

9.1 $f(x) = \begin{cases} 0, & -4 < x < 0 \\ 1/2, & 0 < x < 4 \end{cases}$	9.2 $f(x) = \begin{cases} 3, & 0 < x < 1 \\ -3, & 1 < x < 2 \end{cases}$	9.3 $f(x) = \begin{cases} 0, & -7 < x < 0 \\ 1/3, & 0 < x < 7 \end{cases}$
9.4	9.5	9.6

$f(x) = \begin{cases} 0, & -3 < x < 0 \\ 1/2, & 0 < x < 3 \end{cases}$	$f(x) = \begin{cases} 3, & 0 < x < 2,5 \\ -3, & 2,5 < x < 5 \end{cases}$	$f(x) = \begin{cases} 3, & 0 < x < 1/2 \\ -3, & 1/2 < x < 1 \end{cases}$
9.7 $f(x) = \begin{cases} 0, & -2 < x < 0 \\ 1/5, & 0 < x < 2 \end{cases}$	9.8 $f(x) = \begin{cases} 7, & 0 < x < 2 \\ -7, & 2 < x < 4 \end{cases}$	9.9 $f(x) = \begin{cases} 6, & 0 < x < 1/2 \\ -6, & 1/2 < x < 1 \end{cases}$
9.10 $f(x) = \begin{cases} 0, & -6 < x < 0 \\ 8, & 0 < x < 6 \end{cases}$	9.11 $f(x) = \begin{cases} 4, & 0 < x < 5 \\ -4, & 5 < x < 10 \end{cases}$	9.12 $f(x) = \begin{cases} 3, & 0 < x < 3 \\ -3, & 3 < x < 6 \end{cases}$
9.13 $f(x) = \begin{cases} 3, & 0 < x < 3/2 \\ -3, & 3/2 < x < 3 \end{cases}$	9.14 $f(x) = \begin{cases} 1/3, & 0 < x < 1,5 \\ -1/3, & 1,5 < x < 3 \end{cases}$	9.15 $f(x) = \begin{cases} 9, & 0 < x < 3/2 \\ -9, & 3/2 < x < 3 \end{cases}$
9.16 $f(x) = \begin{cases} 1, & 0 < x < 1/2 \\ -1, & 1/2 < x < 1 \end{cases}$	9.17 $f(x) = \begin{cases} 0, & -5 < x < 0 \\ 8, & 0 < x < 5 \end{cases}$	9.18 $f(x) = \begin{cases} 0, & -7 < x < 0 \\ 3, & 0 < x < 7 \end{cases}$
9.19 $f(x) = \begin{cases} 4, & 0 < x < 3/2 \\ -4, & 3/2 < x < 3 \end{cases}$	9.20 $f(x) = \begin{cases} 1, & 0 < x < 5/2 \\ -1, & 5/2 < x < 5 \end{cases}$	9.21 $f(x) = \begin{cases} 1, & 0 < x < 3/2 \\ -1, & 3/2 < x < 3 \end{cases}$
9.22 $f(x) = \begin{cases} 5, & 0 < x < 2 \\ -5, & 2 < x < 4 \end{cases}$	9.23 $f(x) = \begin{cases} 0, & -4 < x < 0 \\ 9, & 0 < x < 4 \end{cases}$	9.24 $f(x) = \begin{cases} 0, & -3,5 < x < 0 \\ 9, & 0 < x < 3,5 \end{cases}$
9.25 $f(x) = \begin{cases} 0, & -2 < x < 0 \\ 2, & 0 < x < 2 \end{cases}$	9.26 $f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$	9.27 $f(x) = \begin{cases} 0, & -3 < x < 0 \\ 3, & 0 < x < 3 \end{cases}$
9.28 $f(x) = \begin{cases} 0, & -5 < x < 0 \\ 2, & 0 < x < 5 \end{cases}$	9.29 $f(x) = \begin{cases} 1, & 0 < x < 1 \\ -1, & 1 < x < 2 \end{cases}$	9.30 $f(x) = \begin{cases} 2, & 0 < x < 2 \\ -2, & 2 < x < 4 \end{cases}$

Task 10. Using the expansion of the integrand in a power series, calculate the definite integral to within 0,001.

10.1 $\int_0^1 e^{-6x^2} dx$	10.2 $\int_0^1 \frac{dx}{\sqrt[4]{16+x^4}}$	10.3 $\int_0^{2,5} \frac{dx}{\sqrt[3]{125+x^3}}$	10.4 $\int_0^1 \cos x^2 dx$	10.5 $\int_0^{0,2} \frac{1-e^{-x}}{x} dx$
10.6 $\int_0^{0,4} e^{-\frac{3x^2}{4}} dx$	10.7 $\int_0^2 \frac{dx}{\sqrt[3]{64+x^3}}$	10.8 $\int_0^{0,5} \sin(4x^2) dx$	10.9 $\int_0^{0,3} e^{-2x^2} dx$	10.10 $\int_0^{0,2} \cos(25x^2) dx$

10.11 $\int_0^{\frac{4}{5}} \cos\left(\frac{5x}{2}\right)^2 dx$	10.12 $\int_0^{\frac{5}{4}} \frac{dx}{\sqrt[4]{1+x^4}}$	10.13 $\int_1^{\frac{1}{5}} \frac{\ln\left(1+\frac{x}{5}\right)}{x} dx$	10.14 $\int_0^{\frac{4}{3}} \frac{\ln\left(1+\frac{x}{2}\right)}{x} dx$	10.15 $\int_0^{\frac{2}{4}} \frac{dx}{\sqrt[4]{256+x^4}}$
10.16 $\int_0^{\frac{1}{2}} \frac{1-e^{-2x}}{x} dx$	10.17 $\int_0^{\frac{4}{5}} \sin\left(\frac{5x}{2}\right)^2 dx$	10.18 $\int_0^{\frac{5}{3}} \frac{dx}{\sqrt[3]{1+x^3}}$	10.19 $\int_0^{\frac{5}{3}} \frac{dx}{\sqrt[3]{27+x^3}}$	10.20 $\int_0^{\frac{1}{100}} \sin(100x^2) dx$
10.21 $\int_0^{2/5} \frac{dx}{\sqrt{625+x^4}}$	10.22 $\int_0^{0.2} e^{-3x^2} dx$	10.23 $\int_0^{1.5} \frac{dx}{\sqrt[4]{81+x^4}}$	10.24 $\int_0^{\frac{1}{3}} \frac{dx}{\sqrt[3]{8+x^3}}$	10.25 $\int_0^{0.2} \sin(25x^2) dx$
10.26 $\int_0^{0.4} \frac{1-e^{-x/2}}{x} dx$	10.27 $\int_0^{0.5} e^{-\frac{3x^2}{25}} dx$	10.28 $\int_0^{0.1} \frac{\ln(1+2x)}{x} dx$	10.29 $\int_0^{0.5} \cos(4x^2) dx$	10.30 $\int_0^1 \sin x^2 dx$

Task 11. Find the domain of convergence of the series.

11.1 $\sum_{n=1}^{\infty} \frac{2^n x^n}{n^2 + 1}$	11.2 $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$	11.3 $\sum_{n=1}^{\infty} \frac{(x-3)^n}{(n+1)^2 2^{n+1}}$	11.4 $\sum_{n=1}^{\infty} \frac{n x^{n-1}}{2^{n-1} \cdot 3^n}$
11.5 $\sum_{n=1}^{\infty} \frac{(0,1)^n x^{2n}}{n}$	11.6 $\sum_{n=1}^{\infty} \frac{x^{2n}}{8^n}$	11.7 $\sum_{n=1}^{\infty} \frac{n! \cdot x^n}{n^n}$	11.8 $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 2^n}$
11.9 $\sum_{n=1}^{\infty} \frac{x^n}{5^n}$	11.10 $\sum_{n=1}^{\infty} \frac{x^n}{n}$	11.11 $\sum_{n=1}^{\infty} \frac{5^n x^n}{(2n+1)^2 \sqrt{3^n}}$	11.12 $\sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1}$
11.13 $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$	11.14 $\sum_{n=1}^{\infty} \frac{2^n x^n}{\sqrt{n}}$	11.15 $\sum_{n=1}^{\infty} \frac{(-x)^{n+1}}{n^3}$	11.16 $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$
11.17 $\sum_{n=1}^{\infty} \frac{2^n x^n}{2n-1}$	11.18 $\sum_{n=1}^{\infty} (\ln x)^n$	11.19 $\sum_{n=1}^{\infty} \frac{2^n x^n}{\sqrt{2n-1}}$	11.20 $\sum_{n=1}^{\infty} \frac{(n+1)^2 x^n}{2^n}$
11.21 $\sum_{n=1}^{\infty} \frac{x^n}{2^n \sqrt{3n-1}}$	11.22 $\sum_{n=1}^{\infty} \frac{10^n x^n}{\sqrt[n]{n}}$	11.23 $\sum_{n=1}^{\infty} \frac{5^n x^n}{6^n \sqrt[3]{n}}$	11.24 $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} \frac{x^n}{5^n}$
11.25 $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} \frac{x^n}{5^n}$	11.26 $\sum_{n=1}^{\infty} x^n \operatorname{tg} \frac{1}{n}$	11.27 $\sum_{n=1}^{\infty} \frac{3^n x^n}{\sqrt[3]{n}}$	11.28 $\sum_{n=1}^{\infty} x^n \operatorname{tg} \frac{x}{2^n}$
11.29 $\sum_{n=1}^{\infty} (\ln x)^n$	11.30 $\sum_{n=1}^{\infty} (n(n+1)) x^n$	11.31 $\sum_{n=1}^{\infty} \frac{5^n x^n}{\sqrt[3]{n^2}}$	11.32 $\sum_{n=1}^{\infty} x^n \operatorname{tg} \frac{x}{5^n}$

3.3 Solution of an exemplary embodiment

Task 1. For series $\sum_{n=1}^{\infty} \frac{1}{(2n-1) \cdot 3^n}$:

- a) make a general term formula u_n and write the first five terms;
- b) write the n -th partial sum S_n of the series;
- c) write the remaining tail r_n of the series;
- d) check the necessary condition for convergence of the series.

For numerical series $\sum_{n=1}^{\infty} u_n$, where $u_n \in R (n=1, 2, \dots)$,

$S_n = \sum_{k=1}^n u_k$ – the n -th partial sum of the series,

$r_n = \sum_{k=n+1}^{\infty} u_k$ – the remaining tail of the series.

The series converges if there is a finite limit: $S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n u_k$,

otherwise, the series diverges. The number S is called the sum of the series.

If the series converges, then the necessary condition for convergence of the series is performed:

$$\lim_{n \rightarrow \infty} u_n = 0;$$

if $\lim_{n \rightarrow \infty} u_n \neq 0$, then the series diverges.

Solution: a) $u_n = \frac{1}{(2n-1) \cdot 3^n}$;

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1) \cdot 3^n} = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 3^2} + \frac{1}{5 \cdot 3^3} + \frac{1}{7 \cdot 3^4} + \frac{1}{9 \cdot 3^5} + \dots;$$

b) by definition: S_n – the n -th partial sum of the series is equal to the sum of the first n terms of the series, i.e. $S_n = \sum_{k=1}^n u_k$, thus:

$$\sum_{k=1}^n \frac{1}{(2k-1) \cdot 3^k};$$

c) by definition: r_n – the remaining tail of the series is a series obtained from the original one, without the first n terms, i.e. $r_n = \sum_{k=n+1}^{\infty} u_k$, so:

$$r_n = \sum_{k=n+1}^{\infty} \frac{1}{(2k-1) \cdot 3^k};$$

d) $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{(2n-1) \cdot 3^n} = 0$, the necessary condition for convergence of the series is fulfilled, but the conclusion about the convergence of the series cannot be made.

Task 2. Compare with Dirichlet $\sum_{n=1}^{\infty} \frac{1}{n^p}$, i.e. find the parameter p , and investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{n^2 - 1}{(2n-1) \cdot (n^2 + 1)}$.

The second comparison test: consider series $\sum_{n=1}^{\infty} u_n, \sum_{n=1}^{\infty} v_n$.

If $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = A$ (where $A \neq 0, A \neq \infty$) exists, then the series $\sum_{n=1}^{\infty} u_n$ and

$\sum_{n=1}^{\infty} v_n$ simultaneously converge or simultaneously diverge.

For comparison, *the Dirichlet series* is taken: $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$, the convergence of which depends on the parameter p (when $p > 1$ the series converges, when $p \leq 1$ the series diverges). To use the comparison test, we need to find the parameter p and make the appropriate conclusion.

Solution: $u_n = \frac{n^2 - 1}{(2n-1) \cdot (n^2 + 1)}$, then the maximum degree of the numerator

is 2, and the degree of the denominator is 3, then $p = 3 - 2 \leq 1 \rightarrow v_n = \frac{1}{n^1}$.

Consequently, the series $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

As $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{\frac{n^2 - 1}{(2n-1) \cdot (n^2 + 1)}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{(n^2 - 1) \cdot n}{(2n-1) \cdot (n^2 + 1)} = \frac{1}{2} = const$, then

the series $\sum_{n=1}^{\infty} \frac{n^2 - 1}{(2n-1) \cdot (n^2 + 1)}$ also diverges.

Task 3. Investigate the series $\sum_{n=1}^{\infty} \frac{3^n}{(2n+1)!}$ for convergence using the D'Alembert's test.

D'Alembert test can be used if the general term u_n contains exponential function or factorial.

D'Alembert's test: If there is the finite limit of ratio $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = l$, then:

- a) the series converges if $0 \leq l < 1$;
- b) the series diverges if $l > 1$.

If $l = 1$, then the theorem gives no answer on convergence of the series, that is why another test is required.

Solution: for a given series $u_n = \frac{3^n}{(2n+1)!}$, $u_{n+1} = \frac{3^{n+1}}{(2n+3)!}$. On the basis of the

D'Alembert test we have:

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{\frac{3^{n+1}}{(2n+3)!}}{\frac{3^n}{(2n+1)!}} = \lim_{n \rightarrow \infty} \frac{3^{n+1} \cdot (2n+1)!}{(2n+3)! \cdot 3^n} = \lim_{n \rightarrow \infty} \frac{3}{(2n+2) \cdot (2n+3)} = 0 < 1,$$

therefore, the series converges.

Task 4. Investigate for convergence the series $\sum_{n=1}^{\infty} \frac{5^n}{(3n+1)^n}$ using the radical

Cauchy criterion.

The radical Cauchy criterion: If there is the finite limit $\lim_{n \rightarrow \infty} \sqrt[n]{u_n} = l$, then:

- a) the series converges if $0 \leq l < 1$;
- b) the series diverges if $l > 1$.

If $l = 1$, then the theorem gives no answer on convergence of the series, that is why another test is required.

Solution: for a given series we have $u_n = \frac{5^n}{(3n+1)^n}$:

$$\lim_{n \rightarrow \infty} \sqrt[n]{u_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{5^n}{(3n+1)^n}} = \lim_{n \rightarrow \infty} \frac{5}{(3n+1)} = 0 < 1,$$

therefore, the series diverges.

Task 5. Investigate for convergence the series $\sum_{n=1}^{\infty} \frac{1}{(2n+1)\ln(2n+1)}$ using the integral Cauchy test.

Integral Cauchy test: let the terms of the series $\sum_{n=1}^{\infty} u_n$ be positive and not increase, i.e. $u_1 \geq u_2 \geq u_3 \geq \dots$, and $f(x)$ – such a continuous non-increasing function, that $f(1) = u_1, f(2) = u_2, \dots, f(n) = u_n$.

Then the series $\sum_{n=1}^{\infty} u_n$ and the improper integral $\int_1^{\infty} f(x)dx$ simultaneously converge or diverge.

Solution: for $u_n = \frac{1}{(2n+1)\ln(2n+1)} \rightarrow f(x) = \frac{1}{(2x+1)\ln(2x+1)}$

$$\begin{aligned} \int_1^{\infty} f(x)dx &= \int_1^{\infty} \frac{1}{(2x+1)\ln(2x+1)} dx = \left| t = \ln(2x+1) \rightarrow dt = \frac{2}{(2x+1)} dx \right| = \frac{1}{2} \int_{\ln 3}^{\infty} \frac{dt}{t} = \\ &= \frac{1}{2} \ln|t| \Big|_{\ln 3}^{\infty} = \frac{1}{2} (\ln \infty - \ln|\ln 3|) = \infty, \end{aligned}$$

i.e. the improper integral $\int_1^{\infty} \frac{1}{(2x+1)\ln(2x+1)} dx$ diverges. Consequently, the original series $\sum_{n=1}^{\infty} \frac{1}{(2n+1)\ln(2n+1)}$ also diverges.

Task 6. Investigate for conditional or absolute convergence the alternating series: a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)^{\frac{5}{3}}}$; b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[5]{(n+1)^2}}$; c) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(4n-1)}$.

The series of the form

$u_1 - u_2 + u_3 - u_4 + \dots + (-1)^{n+1} u_n + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} u_n$, where $u_n \geq 0 (n = 1, 2, \dots)$

– is called *the alternating series*.

Sign of convergence for alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ according to Leibniz

theorem:

1) $u_1 > u_2 > u_3 > \dots$ – if the terms of the series monotonously decrease;

2) $\lim_{n \rightarrow \infty} u_n = 0$, then the series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converge, and its sum is

positive and less then u_1 .

If the series $\sum_{n=1}^{\infty} |u_n|$ converges, then the series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ is called

absolutely convergent; if the series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges, but the series $\sum_{n=1}^{\infty} |u_n|$

diverges, then the series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ is called *conditionally convergent*.

Therefore, investigating conditional or absolute convergence, three cases are possible: the series absolutely converges; or the series conditionally converges; or the series diverges.

Solution:

a) investigate the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)^{\frac{5}{3}}}$ for absolute convergence, where series of

absolute values $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{(n+2)^{\frac{5}{3}}} \right| = \sum_{n=1}^{\infty} \frac{1}{(n+2)^{\frac{5}{3}}}$ converges (because $p = \frac{5}{3} > 1$).

Consequently the original series $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)^{\frac{5}{3}}}$ absolutely converges.

b) consider the series of absolute values for the given series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[5]{(n+1)^2}}$:

$\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt[5]{(n+1)^2}} \right| = \sum_{n=1}^{\infty} \frac{1}{(n+1)^{\frac{2}{5}}}$, which when compared with the Dirichlet

series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ($p = \frac{2}{5} < 1$) diverges. Consequently, the original series cannot converge absolutely.

Let us verify the fulfillment of the Leibniz conditions:

$$1) \frac{1}{\sqrt[5]{2^2}} > \frac{1}{\sqrt[5]{3^2}} > \frac{1}{\sqrt[5]{4^2}} > \dots; \quad 2) \lim_{n \rightarrow \infty} \frac{1}{\sqrt[5]{(n+1)^2}} = 0.$$

They are satisfied, therefore, the original series converges conditionally.

c) for the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(4n-1)}$ let us verify the fulfillment of the Leibniz conditions:

$$1) \frac{1}{3} > \frac{2}{7} > \frac{3}{11} > \dots; \quad 2) \lim_{n \rightarrow \infty} \frac{n}{4n-1} = \frac{1}{4} \neq 0,$$

where for the initial series the second Leibnitz condition is not satisfied. Consequently, the original series diverges.

Task 7. Find the radius and interval of convergence of the series:

$$a) \sum_{n=1}^{\infty} \left(\frac{n+4}{2n} \right)^n (3x-5)^n; \quad b) \sum_{n=1}^{\infty} \frac{n^2+1}{5^n} (5x+1)^n.$$

The radius of convergence of a power series $\sum_{n=1}^{\infty} a_n (x-x_0)^n$ is found in one of two formulas:

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \quad \text{or} \quad R = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{|a_n|}}.$$

The interval of convergence of the series $\sum_{n=1}^{\infty} a_n (x-x_0)^n$ is defined by the inequality: $|x-x_0| < R$.

Solution:

a) let's bring this series to a standard form:

$$\sum_{n=1}^{\infty} \left(\frac{n+4}{2n} \right)^n (3x-5)^n = \sum_{n=1}^{\infty} \left(\frac{n+4}{2n} \right)^n \cdot 3^n \cdot \left(x - \frac{5}{3} \right)^n.$$

For the resulting series: $a_n = \left(\frac{n+4}{2n} \right)^n \cdot 3^n = \left(\frac{3 \cdot (n+4)}{2n} \right)^n$, $x_0 = \frac{5}{3}$.

Therefore, the radius of convergence of the series is found by the formula:

$$R = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{|a_n|}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{\left(\frac{3 \cdot (n+4)}{2n} \right)^n}} = \lim_{n \rightarrow \infty} \frac{2n}{3 \cdot (n+4)} = \frac{2}{3}.$$

The interval of convergence of the series:

$$\begin{aligned} \left| x - \frac{5}{3} \right| &< \frac{2}{3} \Rightarrow -\frac{2}{3} < x - \frac{5}{3} < \frac{2}{3} \Rightarrow \\ -\frac{2}{3} + \frac{5}{3} &< x < \frac{2}{3} + \frac{5}{3} \Rightarrow 1 < x < \frac{7}{3}; \end{aligned}$$

b) let's bring the given series $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n} (5x+1)^n$ to a standard form:

$$\sum_{n=1}^{\infty} \frac{n^2+1}{5^n} (5x+1)^n = \sum_{n=1}^{\infty} \frac{(n^2+1) \cdot 5^n}{5^n} \left(x + \frac{1}{5} \right)^n = \sum_{n=1}^{\infty} (n^2+1) \left(x + \frac{1}{5} \right)^n.$$

For the resulting series: $a_n = (n^2+1)$, $x_0 = -\frac{1}{5}$.

Therefore, the radius of convergence and the interval of convergence of the series we find:

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{n^2+1}{(n+1)^2+1} = \lim_{n \rightarrow \infty} \frac{n^2+1}{n^2+2n+2} = \left| \frac{\infty}{\infty} \right| = \left| \frac{n^2}{n^2} \right| = 1;$$

$$\left| x + \frac{1}{5} \right| < 1 \Rightarrow -1 < x + \frac{1}{5} < 1;$$

$$-1 - \frac{1}{5} < x < 1 - \frac{1}{5} \Rightarrow -\frac{6}{5} < x < \frac{4}{5}.$$

Task 8. Calculate the sum of the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2 2^n}$ with precision $\alpha=0,001$.

Solution: since the series is alternating and converging, the n -th partial sum of the series S_n is an approximation to its sum S with absolute error α :

$$|S - S_n| = |r_n| < \alpha \rightarrow S = \sum_k a_k \approx S_n .$$

We calculate several consecutive first terms of this series: $a_1 = 0.5$, $a_2 = -0.0625, \dots a_n > \alpha$, i.e. until we get the term a_{n+1} , whose absolute value is less than the specified accuracy. In our case it is:

$$a_{n+1} = a_6 = \frac{1}{6^2 2^6} = 0.000434 < \alpha.$$

$$S = S_5 = 0.5 - 0.0625 + \frac{1}{75} - \frac{1}{256} + \frac{1}{800} \approx 0.049.$$

Task 9. Expand in Fourier series periodic function:

$$f(x) = \begin{cases} -1, & 0 \leq x < 2 \\ 2, & 2 \leq x \leq 4 \end{cases}.$$

Solution: to determine the Fourier coefficients we use the formulas:

$$\begin{aligned} a_0 &= \frac{1}{l} \int_a^{a+2l} f(x) dx; \\ a_n &= \frac{1}{l} \int_a^{a+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx; \\ b_n &= \frac{1}{l} \int_a^{a+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx, \end{aligned}$$

where $a=0$, $a+2l=4$, then the half period is $l=(4-0)/2=2$.

We obtain:

$$\begin{aligned} a_0 &= \frac{1}{2} \int_0^4 f(x) dx = \frac{1}{2} \int_0^2 (-1) dx + \frac{1}{2} \int_2^4 (2) dx = -\frac{1}{2}(2-0) + \frac{1}{2}(4-2) = 1; \\ a_n &= \frac{1}{2} \int_0^4 (-1) \cos\left(\frac{n\pi x}{2}\right) dx + \frac{1}{2} \int_2^4 (2) \cos\left(\frac{n\pi x}{2}\right) dx = \\ &= -\frac{1}{2\pi n} \sin\left(\frac{\pi n x}{2}\right) \Big|_0^2 + \frac{1}{\pi n} \sin\left(\frac{\pi n x}{2}\right) \Big|_2^4 = 0; \end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{2} \int_0^2 (-1) \sin\left(\frac{n\pi x}{2}\right) dx + \frac{1}{2} \int_2^4 (2) \sin\left(\frac{n\pi x}{2}\right) dx = \\
&= \frac{2}{2\pi n} \cos(\pi nx) \Big|_0^2 - \frac{4}{2\pi n} \cos(\pi nx) \Big|_2^4 = \\
&= \frac{1}{2\pi n} (\cos(\pi n) - \cos 0) - \frac{1}{\pi n} (\cos 2\pi n - \cos \pi n) = \frac{3}{2\pi n} ((-1)^n - 1).
\end{aligned}$$

Substituting the obtained coefficients in the formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right),$$

where $l = 2$, we obtain the desired expansion:

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{3}{2\pi n} ((-1)^n - 1) \left(\sin\left(\frac{\pi nx}{2}\right)\right).$$

Task 10. Using the expansion of the integrand in a power series, calculate the

definite integral $\int_0^{1/2} \frac{1}{\sqrt{1+x^4}}$ to within 0,00001.

Solution: decompose the integrand in the binomial series:

$$(1+t)^m = 1 + mt + \frac{m(m-1)}{2!} t^2 + \frac{m(m-1)(m-2)}{3!} t^3 + \dots + \frac{m(m-1)\dots(m-n+1)}{n!} t^n + \dots$$

taking a replacement $t = x^4$, $m = -\frac{1}{2}$ in it:

$$\frac{1}{\sqrt{1+x^4}} = (1+x^4)^{-\frac{1}{2}} = 1 - \frac{1}{2}x^4 + \frac{1}{2!} \frac{1 \cdot 3}{2 \cdot 2} x^8 - \frac{1}{3!} \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2} x^{12} + \dots,$$

this series converges for $|x| < 1$. We integrate this series in the range from 0 to 0.5:

$$\begin{aligned}
\int_0^{0.5} \frac{1}{\sqrt{1+x^4}} dx &= \int_0^{0.5} \left(1 - \frac{1}{2}x^4 + \frac{1}{2!2\cdot2}x^8 - \frac{1}{3!2\cdot2\cdot2}x^{12} + \dots \right) dx = \\
&= \left. \left(x - \frac{1}{2}\frac{x^5}{5} + \frac{1}{2!2\cdot2}\frac{x^9}{9} - \frac{1}{3!2\cdot2\cdot2}\frac{x^{13}}{13} + \dots \right) \right|_0^{0.5} = \\
&= \frac{1}{2} - \frac{1}{2\cdot5\cdot2^5} + \frac{1}{2!2\cdot2}\frac{1}{2^9\cdot9} - \frac{1}{3!2\cdot2\cdot2}\frac{1}{13\cdot2^{13}} + \dots
\end{aligned}$$

We received an alternating series, the sum of which must be found with an accuracy of 0,0001. We calculate several consecutive first terms of this series: $a_1 \approx 0.5$; $a_2 \approx -0.00313$; $a_3 \approx 0.00008$.

As $|a_3| < 0.00001$ then, according to the property of alternating convergent series, we have:

$$\int_0^{0.5} \frac{1}{\sqrt{1+x^4}} dx \approx \frac{1}{2} - \frac{1}{2\cdot5\cdot2^5} = \frac{1}{2} - \frac{1}{2\cdot5\cdot2^5} = 0.5 - 0.00313 = 0.4969.$$

Task 11. Find the domain of convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}5^n} (5x+1)^n$.

Solution: first, the radius and convergence interval are calculated:

$$a_n = \frac{1}{\sqrt{n}5^n}; \quad a_{n+1} = \frac{1}{\sqrt{n+1}5^{n+1}};$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \sqrt{\frac{n+1}{n}} 5 = 5 \cdot \lim_{n \rightarrow \infty} \sqrt{\frac{n+1}{n}} = \left| \frac{\infty}{\infty} \right| = 5 \cdot \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n}} = 5,$$

$$|5x+1| < 5 \rightarrow -5 - 1 < 5x < 5 - 1 \rightarrow -\frac{6}{5} < x < \frac{4}{5};$$

a) when $x = \frac{4}{5}$ we get a numerical series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is investigated for

convergence on the basis of comparison ($p = \frac{1}{2}$) with Dirichlet series: $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is

divergent series. Therefore, the value $x = \frac{4}{5}$ is not included in the domain of convergence of the series;

6) when $x = -\frac{6}{5}$ we get a numerical series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. It converges

conditionally, since the conditions of Leibniz are satisfied:

$$1) \frac{1}{\sqrt{1}} > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \dots$$

$$2) \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

Consequently, the value $x = -\frac{6}{5}$ enters the region of convergence of the series.

Thus, $[-\frac{6}{5}; \frac{4}{5})$ is the domain of convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} 5^n} (5x+1)^n.$$

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