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**COMPUTER MODELLING METHODS  
AND APPLICATIONS**

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The tutorial is intended for the systematic study of the principles and methods of computer modeling of processes occurring in complex systems, its mathematical apparatus, typical mathematical schemes and the use of computer simulation results in various areas of human activity.

The book consists of thirteen chapters, combined in two parts. The first part gives an account of the mathematical apparatus of methods for simulating random events, quantities, processes and flows, as well as algorithms and programs for their implementation. In the second part, the methodological principles of organizing computer modeling are considered and typical schemes used in modeling various systems and specific models for a wide class of real objects are presented.

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## Foreword

Computer modeling is currently an intensively developing area of computer science, based on the use of simulation models, and has become widespread in the economy, industry, ecology, subsoil use and other areas of human activity.

This book is intended to study the principles and methods of computer modeling of processes occurring in complex systems, its mathematical apparatus and typical mathematical schemes and specific examples of implementations.

This book was formed as a result of a generalization of the author's scientific experience in the field of modeling and process optimization in complex systems and for several years I gave lectures to senior students of leading technical universities in Kazakhstan and consists of an introduction and 13 chapters, combined in two parts.

The introduction provides a classification of modeling methods, describes the purpose and capabilities of computer modeling and a vivid example illustrating its essence.

The first part provides a systematic presentation of the mathematical apparatus of methods for simulating random events, variables, processes and flows, and their implementation algorithms.

In the second part, there were considered the methodological principles of computer modeling organization, typical schemes used in modeling various complex systems and were presented the results of computer modeling for a wide class of real objects based on scientific research by the authors.

The book is intended for specialists, students, undergraduates and doctoral students interested in using computers in scientific and applied researches.

## Introduction

### B.1 Modeling methods classification

The models used in science, technology and economics can be divided into two types: physical and mathematical. Physical modeling is understood as the reproduction of the process under study with the preservation of its physical nature. Consequently, the physical model is a real system that differs from the modeled system in dimensions and other parameters, but at the same time retains the properties of the original that are important for research. Here are some examples of physical models. Models of the solar system installed in planetariums clearly demonstrate the changing seasons, lunar and solar eclipses and other astronomical phenomena. A laboratory installation for producing a certain product can be considered as a model of industrial production of this product. A separate enterprise, placed in special economic conditions, can be considered as a model that serves to verify any proposals in the field of planning and management. Physical models are usually very concrete and specific, give reliable and visual results. However, physical models are difficult to manipulate for experimental purposes, they are complex and quite expensive and, therefore, have a limited implementation area. Mathematical modeling has more advanced features. Mathematical modeling is understood as a method of studying various processes by studying events having different physical contents, but described by the same mathematical relationships. Thus, a system of linear equations and inequalities can serve as a model for planning the activities of an enterprise or organizing transportation. Due to its versatility and relative simplicity, mathematical models have become very widespread in various researches. However, over time, there are appeared problems for which the existing mathematical schemes are insufficient. The reason for this is the transition to the study of systems consisting of a large number of interacting objects. An essential feature of these systems is the complexity of the interaction nature of their constituent objects, the need to take into account the influence of various disturbing factors and the dynamism of the processes taking place in these systems. These reasons led to the occurrence of a new direction in modeling, called "simulation or computer modeling". Under computer modeling we mean a computer simulation of the process of functioning of a probabilistic model of a certain object in order to evaluate its required characteristics [1].

The scope of computer modeling is the study of systems consisting of a large number of interacting objects. Currently, this method is used for researches in many areas, including:

- production management;
- automatic telephone exchange service systems design;
- traffic regulation;
- warehouse management;
- the functioning of military equipment tools, etc.

## **B.2 The content and capabilities of computer modeling**

The essence of the considered modeling method is to implement on the computer a special (modeling) algorithm that imitates the behavior and interaction of the elements of a complex system, taking into account random disturbing factors. The influence of random factors on the process is reproduced using random numbers with probabilistic characteristics given or generated during the modeling process.

The modeling algorithm approximately reproduces the original process in the sense of its functioning in time, and the elementary phenomena that make up the process are simulated while maintaining their logical structure and sequence of flow over time [2]. In addition, this algorithm allows, based on the initial data containing information about the initial state of the process and its parameters, to obtain information about the states of the process at arbitrary points in time.

Computer simulation is applicable to the study of very complex processes and has significant advantages over other types of modeling. Its main advantage is the ability to solve problems of considerable complexity: the system under study can simultaneously contain elements of continuous and discrete action, to be the subject of influence of numerous random factors of complex nature, can be described by very cumbersome relations, etc.

In addition, computer simulation does not require the creation of special equipment for each new task and makes it easy to change the values of the parameters of the systems under study and the initial conditions.

A significant role is played by computer modeling in solving problems related to control automation. The simulation results allow revealing the laws of the process that are significant from the point of view of automated control, to determine the flows of control information and to reasonably choose control algorithms. Using computer modeling, the effectiveness of various control principles, design variants for control systems, as well as the performance and reliability of control equipment can be evaluated.

Along with the noted advantages, computer modeling, like any other numerical method, has the significant disadvantage that the solution obtained is always private in nature, corresponding to fixed values of the system parameters, input information and initial conditions [3].

Despite this very serious disadvantage, computer modeling is currently the most effective means of studying complex systems, and sometimes the only practically accessible method of obtaining information on the behavior of a system that interests us, especially at the stage of its design and modernization.

## **B.3 The simplest example of computer modeling**

In order to understand the essence of computer modeling, let examine a simple example. Let consider the work of a shoe shiner at a busy intersection, for example, for one hour from 9.00 to 10.00. Suppose that the time intervals between

successive customer appearances are evenly distributed in the interval from 1 to 10 minutes (for simplicity, we will round the time to the nearest whole number of minutes). Suppose further that the time required to service each client is evenly distributed in the interval from 1 to 6 minutes. We are interested in the average time that the client spends on cleaning shoes (including both waiting and service), and the percentage of time that the cleaner is not busy. In order to simulate the work of a shoe shiner, it is necessary to set up an artificial experiment that reflects the basic conditions of the situation. To do this, you need to come up with a way to simulate the artificial sequence of customer arrival and the time required to service each of them [4]. One of the methods that could be applied here is to take ten numbered from 1 to 10 planks and one cube from a set for a first grader. Then put these planks in an empty box and, shaking it, mix. By pulling the planks out of the box and reading out the drawn number, you can get the time intervals between the appearance of the previous and subsequent customers. Throwing a cube and reading the number of points from its upper side, we can imagine the time of servicing each client. Repeating these operations in the indicated sequence (returning the planks each time and shaking the box before each pulling), we can obtain time series representing the time intervals between successive moments of customers arrival and the corresponding duration of service. The table. B.1 shows the results of modeling the work of a shoe cleaner for one hour. According to these data, it is possible to obtain, for example, the average time spent by a client for cleaning shoes  $t_{cp} = (44+14)/ 12 = 4,83$  min and the percentage of non-productive time of a shoe shiner equal to 27%.

This simple example clearly demonstrates the role of various random patterns in computer modeling.

Table B.1

Client No.	The duration of the time interval between clients, min	Duration of customer service, min	Customer arrival time	Service start time	Duration of customer waiting for service start, min	Service end time	Shoe shiner idle duration, min
1	0	2	9.00	9.00	0	9.02	0
2	8	3	9.08	9.08	0	9.11	6
3	1	6	9.09	9.11	2	9.17	0
4	6	4	9.15	9.17	2	9.21	0
5	3	6	9.18	9.21	3	9.27	0
6	9	1	9.27	9.27	0	9.28	0
7	8	3	9.35	9.35	0	9.38	7
8	2	5	9.37	9.38	1	9.43	0
9	1	2	9.38	9.43	5	9.45	0
10	7	4	9.45	9.45	0	9.49	0
11	3	2	9.48	9.49	1	9.51	0
12	6	6	9.54	9.54	0	10.00	3
Total		44			14		16

# PART 1. RANDOM PATTERNS MODELING

## Chapter 1. Random numbers modeling

### 1.1 Random numbers and the principle of their modeling

During the study of complex systems using computer modeling, there are widely used random events, random variables with given laws of probability distribution, and various random processes. The main method of computer simulation of random laws of any nature is reduced to modeling a sequence of random numbers with a uniform distribution law on the interval  $[0, 1]$  and to its subsequent functional transformation.

The choice as the initial or basic sequence of random numbers that are implementations of  $z_i$  of a random variable  $\xi$  uniformly distributed on the interval  $[0, 1]$  is caused by the following two factors:

- 1) almost from the first days of the computers creation scientists became interested in the problem of random numbers modeling with a uniform distribution and developed sufficiently effective algorithms for their simulation;
- 2) uniform distribution is the simplest among random patterns and is easily amenable to mathematical transformations.

The random variable  $\xi$  obeys the uniform distribution law on the interval  $[a, b]$  if its distribution density function is a continuous function constant on the interval  $[a, b]$  and equal to zero outside this interval.

$$f(z) = \begin{cases} 1/(b-a) & \text{at } z \in [a, b], \\ 0 & \text{at } z \notin [a, b]. \end{cases}$$

The distribution function of the random variable  $\xi$  has the form:

$$F(z) = \begin{cases} 0 & \text{at } z < a, \\ (z-a)/(b-a) & \text{at } a \leq z \leq b, \\ 1 & \text{at } z > b. \end{cases}$$

The mathematical expectation, dispersion and standard deviation, respectively, are:

$$m_z = \int_a^b z \cdot f(z) dz = \frac{a+b}{2}; \quad \sigma_z^2 = \int_a^b (z-m)^2 \cdot f(z) dz = \frac{(b-a)^2}{12}; \quad \sigma_z = \frac{b-a}{2\sqrt{3}}.$$

In the particular case, when the variable  $\xi$  is uniformly distributed on the interval  $[0, 1]$ , we have:

$$f(z) = \begin{cases} 1 & \text{at } z \in [0,1]; \\ 0 & \text{at } z \notin [0,1]; \end{cases}$$

$$F(z) = \begin{cases} 0 & \text{at } z < 0; \\ z & \text{at } 0 \leq z \leq 1; \\ 1 & \text{at } z > 1; \end{cases}$$

$$m_z = 1/2, \quad \sigma_z^2 = 1/12, \quad \sigma_z = 1/2\sqrt{3}.$$

Taking into account the important role for modeling various random patterns of the variable with a uniform distribution on the interval  $[0, 1]$ , let consider several methods for its computer simulation. All these methods are based on recurrence relations and produce pseudo-random numbers.

*Definition.* Pseudorandom numbers are called  $z$  implementations of a random variable  $\xi$ , obtained using mathematical expressions, in particular, recurrence relations.

Obviously, the probabilistic properties of pseudo-random numbers may differ from the properties of ideal random numbers. Therefore, during the development of methods for pseudo-random numbers modeling, rather stringent requirements are imposed on them. Thus, sequences obtained using good methods should consist of uniformly distributed, statistically independent, reproducible and non-repeating numbers [5]. In addition, these methods should work quickly and occupy a minimum amount of memory. If these requirements are met, the difference between pseudo-random numbers and purely random numbers can be neglected.

Most of the methods used in practice to obtain pseudo-random numbers are based on first order recurrence relations:

$$z_{i+1} = \Phi(z_i), \tag{1.1}$$

where the number  $z_0$  is set. But there are certain requirements for this formula.

Indeed, an arbitrary function of the form (1.1) cannot generate a “good” sequence of pseudo-random numbers, since the points with coordinates  $\{z, \Phi(z)\}$  are not evenly distributed over the entire surface of the rectangle (Fig. 1.1), but lie on the curve  $\Phi(z)$ . Therefore, a “good” sequence of pseudo-random numbers can be generated only by a function whose graph fills the unit square very densely. As an example, we can give a function (Figure 1.2):

$$z_{n+1} = D(g \cdot z_n),$$

where  $D$  – fraction allocation operation,  $g$  – quite a large number [4].

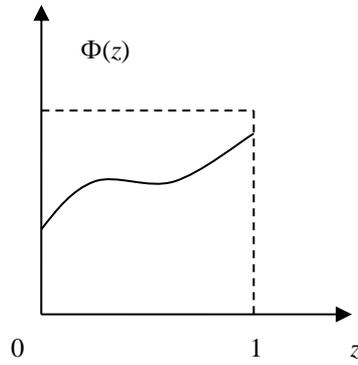


Figure 1.1 - Arbitrary function graph

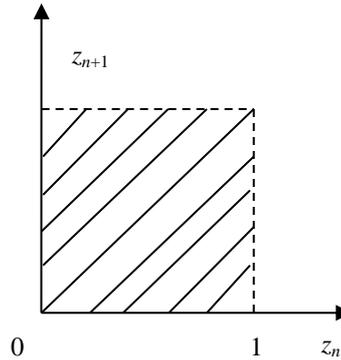


Figure 1.2 - Graph of the fractional part allocation function

The above condition is only necessary, but not enough, for the formula (1.1) to generate “good” pseudo-random numbers. Indeed, at first, the form of the function  $\Phi(z)$  was allocated as complex and difficult to understand as possible, for example [12]:

$$\Phi(z) = 10^{-2k} U[10^{2k} D(10^k z^2)], \quad (1.2)$$

where  $U$  – operation of the whole part allocation.

But the lack of theory regarding the selection of the type of function led to adverse consequences. For example, the next sequence number obtained using such a function could sometimes, and completely unpredictably, go to zero, and then all subsequent elements of it would also be zero. Often the sequences had a small value of the numbers repetition period. Therefore, since the late forties, scientists during the selection of the type of function began to use the number theory apparatus. This made it possible to know in advance the length of the period of the sequence of pseudo-random numbers and to ensure the specified quality of the resulting numbers. Let consider some of the most famous methods for pseudo-random numbers modeling. Moreover, further we will use the term "random numbers" instead of the term "pseudo-random numbers", since we assume that most of the application algorithms proposed below model sequences with sufficiently good statistical properties.

## 1.2 Truncation method

This method is based on the fact that the digit of the next random number are obtained by discarding or "truncating" part of the digits in the result of a nonlinear transformation over one or more previous numbers.

The first algorithm for modeling of evenly distributed random numbers using the idea of truncation was proposed by von Neumann and Metropolis in 1946. This algorithm is called the "middle squares" and operates with  $2k$ -digit numbers [18].

The computational procedure of the algorithm consists of the following steps:

Step 1. Accept  $z_i = 0, a_1, a_2, \dots, a_{2k}$ .

Step 2. Square  $z_i$   $z_i^2 = 0, b_1, b_2, \dots, b_k, b_{k+1}, \dots, b_{3k}, b_{3k+1}, \dots, b_{4k}$ .

Step 3. Select the average  $2k$  digits of the obtained square and consider them as digits of the next implementation of the sequence  $z_{i+1} = 0, b_{k+1}, b_{k+2}, \dots, b_{3k}$ .

It is easy to verify that this algorithm corresponds to the function (1.2) already familiar to us.

*Example 1.2.1.* Let  $z_0 = 0,1981$  and, respectively,  $k=2$

Solution:

$$\begin{array}{ll} z_0=0,1981 & z_0^2=0,03924361 \\ z_1=0,9243 & z_1^2=0,85433049 \\ z_2=0,4330 & z_2^2=0,18748900 \\ z_3=0,7489 & \text{and etc.} \end{array}$$

Unfortunately, the middle squares algorithm in many cases does not give statistically satisfactory results. The generated sequence has more than necessary numbers with small values, i.e.  $m_z < 0$ ; often degeneration of the sequence goes to zero. So, in a series of experiments conducted by the American J. Forsythe in the early fifties, the following results were obtained. Among 16 initial values, 12 ones led to sequences ending in a cycle: 0,6100; 0,2100; 0,4100; 0,8100; 0,6100, and two values - to the degeneration of the sequence.

Sometimes in the sequence developed by the middle squares algorithm there is no randomness at all [34].

*Example 1.2.2.* Let  $z_0=0,4500$

Solution:

$$\begin{array}{ll} z_0=0,4500 & z_0^2=0,20250000 \\ z_1=0,2500 & z_1^2=0,06250000 \\ z_2=0,2500 & z_2^2=0,06250000 \\ z_3=0,2500 & \text{and etc.} \end{array}$$

Currently, due to these disadvantages, the middle squares algorithm is not widely used and it has only historical interest for us. Its former popularity was explained by simplicity and originality.

The followers of John von Neumann proposed various modifications of this algorithm. Significantly better results are obtained, for example, by an algorithm based on a function [5].

$$\Phi(z_n, z_{n-1}) = 10^{-2k} U[10^{2k} D(10^k z_n z_{n-1})]. \quad (1.3)$$

But, nevertheless, at present, almost all standard library programs for modeling sequences of uniformly distributed random numbers implement the methods of residue and summation.

### 1.3 Residue method (congruential method)

This method was proposed by D. Lemer in 1948 [18] and in general case is based on a linear formula of the following form:

$$z_{n+1}^* = az_n^* + C(\text{mod } m), \quad (1.4)$$

where  $z_0^*$ ,  $a$ ,  $c$  and  $m$  – non-negative integers.

The notation (1.4) means that  $z_{n+1}^*$  is the remainder obtained by dividing the expression  $az_n^* + C$  by  $m$  or, in other words,  $z_{n+1}^*$  is the smallest positive residue of  $az_n^* + C$  by the module  $m$ . Formula (1.4) at any values of its parameters can give only a finite set of integer random numbers, after which the sequence begins to repeat. This follows from the obvious restriction  $P < m$ , where  $P$  is the period of the sequence.

*Example 1.3.* Let  $a=7$ ,  $c=z_0=5$ ,  $m=9$ .

Then:

$$z_1^* = 5 * 7 + 5(\text{mod } 9) = 4,$$

$$z_2^* = 4 * 7 + 5(\text{mod } 9) = 6,$$

$$z_3^* = 6 * 7 + 5(\text{mod } 9) = 2,$$

$$z_4^* = 2 * 7 + 5(\text{mod } 9) = 1,$$

$$z_5^* = 1 * 7 + 5(\text{mod } 9) = 3,$$

$$z_6^* = 3 * 7 + 5(\text{mod } 9) = 8,$$

$$z_7^* = 8 * 7 + 5(\text{mod } 9) = 7,$$

$$z_8^* = 7 * 7 + 5(\text{mod } 9) = 0,$$

$$z_9^* = 0 * 7 + 5(\text{mod } 9) = 5$$

and etc.

A special case of expression (1.4) is the formula:

$$z_{n+1}^* = az_n^* (\text{mod } m), \quad (1.5)$$

obtained at  $c=0$ .

This formula simulates random sequences somewhat faster, but with a relatively shorter period.

*Example 1.4.* The values of the parameters  $a$ ,  $m$  and the initial number  $z_0^*$  are kept unchanged. Then:

$$z_1^* = 5 * 7 (\text{mod } 9) = 8,$$

$$z_2^* = 8 * 7 (\text{mod } 9) = 2,$$

$$z_3^* = 2 * 7 \pmod{9} = 5,$$

$$z_4^* = 5 * 7 \pmod{9} = 9.$$

Example 1.3 illustrates one important advantage of the residue method. Namely, using expression (1.4), the residue method eliminates the degeneration of a sequence of random numbers. At the same time, these examples show that with an arbitrary selection of parameter values in formulas (1.4) and (1.5), we will not get a sequence of random numbers with good statistical properties. Therefore, the parameters  $a$ ,  $c$ ,  $m$  and the initial value  $z_0^*$  must be chosen so as to ensure the maximum period of the sequence, the maximum speed of its generation and the minimum correlation between the simulated numbers.

It was established [12] that for the numerical implementation of the residue method the formula (1.5) is convenient, in which  $m = 2^b$ , where  $b$  - the number of binary digits in the machine word. Moreover, the maximum period of the sequence, equal to  $P = m/4$ , can be obtained if the following conditions are satisfied:

- 1)  $z_0^*$  - any positive odd integer;
- 2)  $a = 8t \pm 3$ , where  $t$  - any positive number.

During the modeling a sequence of random numbers with the help of the formula (1.4), we can achieve the maximum period length equal to  $m$ . This result follows from theorem 1.1. The length of the period of the sequence obtained by the formula (1.4) is  $m$  if and only if:

- a)  $c$  and  $m$  are mutually prime numbers;
- b)  $a-1$  is a multiple of  $r$  for any prime  $r$  that is a divisor of  $m$ ;
- c)  $a-1$  is a multiple of 4 if  $m$  is a multiple of 4.

A proof of this theorem can be found in [12].

Since the period has a length  $m$ , each number from 0 to  $m-1$  will occur exactly once in the simulated sequence. Therefore, in this case, the choice of  $z_0^*$  does not affect the length of the period.

In conclusion, it should be noted that according to formulas (1.4) and (1.5), sequences of random numbers with a uniform distribution on the interval  $[0, m]$  are modeled, and in order to obtain numbers on the interval  $[0, 1]$ , these formulas must be supplemented by the expression:

$$z_{n+1} = m^{-1} z_{n+1}^* \quad (1.6)$$

#### 1.4 Summation method

The summation method was proposed by Van Weingarden and uses a general linear formula of the form [5]:

$$z_{k+j} = \sum_{l=0}^{k-1} a_l z_{j+l} + C \pmod{m}, \quad j = 1, 2, \dots, \quad (1.7)$$

This method allows to obtain longer periods in a sequence of uniformly distributed random numbers compared to the residue method, since here the coincidence of individual members of the sequence does not lead to the coincidence

of all its subsequent elements. The latter is possible only if the entire set of numbers involved in formula (1.7) is repeated. In addition, the sequence numbers obtained by the summation method have a small correlation.

The simplest dependence (1.7) can be obtained by accepting:

$$a_2 = a_3 = \dots = a_{k-1} = c = 0, \quad a_0 = a_1 = 1.$$

Then:

$$z_{j+1} = z_j + z_{j-1} \pmod{m}. \quad (1.8)$$

This expression was called the Fibonacci formula and had a fairly wide application in the early fifties. However, the numbers generated by the Fibonacci formula were not random enough. The only exception was the Davis algorithm, which, by a successful selection of the first two initial values  $z_0$  and  $z_1$ , managed to obtain a good sequence of uniformly distributed random numbers.

The Davis algorithm uses the formula [20]:

$$z_{j+1} = z_j + z_{j-1} \pmod{4}, \quad (1.9)$$

where  $z_0 = \pi$ ,  $z_1 = 5^{17} 2^{-42} = 0,542101887$ .

Currently, the most widely used among the summation method algorithms are additive algorithms using the formula:

$$z_{j+1} = z_j + z_{j-k} \pmod{m},$$

where  $k$  – quite a large number ( $k > 16$ ). With the appropriate set  $z_0, z_1, \dots, z_k$  this formula is a good source of random numbers.

### **1.5 Experimental determination of the period and length of the aperiodicity segment of a random sequence**

As follows from the foregoing, a characteristic feature of all random number modeling methods is that when they are implemented on a computer, they generate periodic sequences. Indeed, in the code of any computer, only a finite number of all numbers on the interval  $[0, 1]$  can be written down, so sooner or later some value of  $z_L$  coincides with one of the previous values of  $z_l$ . Then the following equality holds:

$$z_{L+i} = z_{l+i}, \quad i = 1, 2, \dots \quad (1.10)$$

Let  $L$  – smallest number satisfying (1.10) at some  $l$  ( $l < L$ ). Set of numbers  $z_0, z_1, \dots, z_{L-1}$  (Fig.1.3) is called *aperiodicity segment of a sequence*, the number  $L$  – *aperiodicity segment length*. It is obvious that  $L-l$  is equal to the period  $P$ .

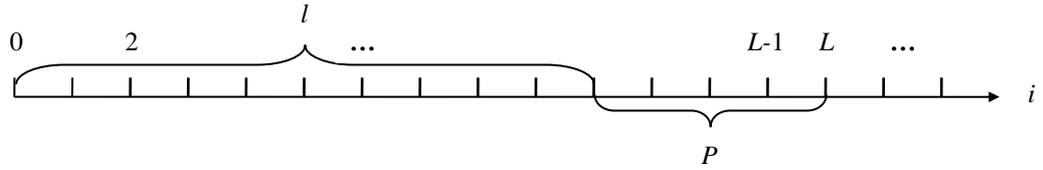


Figure 1.3 - Aperiodicity segment of a sequence

This figure clearly shows that the sequence will be random if the number of its elements does not exceed  $L$ . Therefore, it is necessary to be able to find the exact value of  $L$  and  $P$ . One way to experimentally define the values of  $L$  and  $P$  is as follows [29, 36]. Let  $\{z_i\}$ - a sequence of random numbers obtained using one of the above algorithms. Let fix an element  $z_N$  of this sequence, where  $N$  is obviously larger than the difference  $L-P$  (Figure 1.4):

$$z_n = z_k \quad k > N.$$

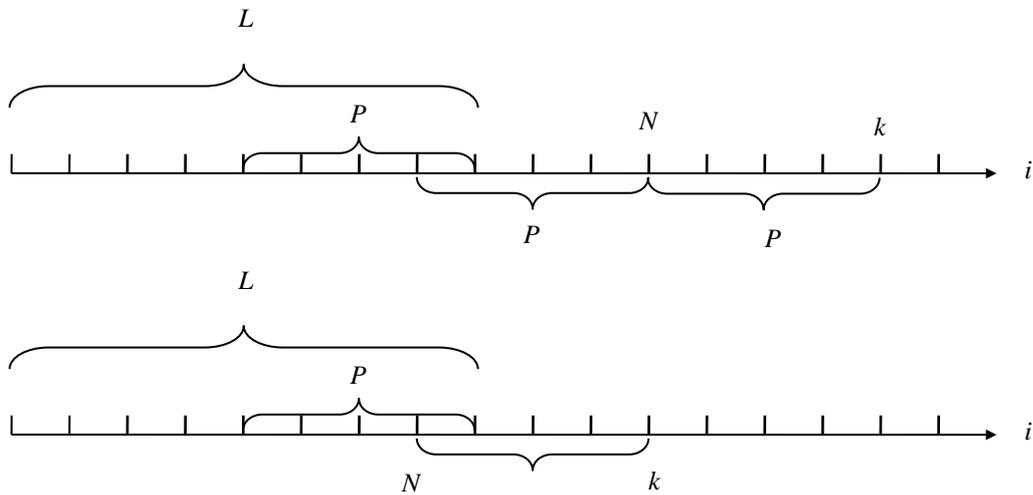


Figure 1.4 - The values of  $L$  and  $P$  in a sequence of random numbers

Next, compare  $z_N$  with the numbers  $z_{N+1}, z_{N+2}, \dots$  until they coincide with any of the following members of the sequence.

Let  $z_n$  be equal to  $z_k$  ( $k > N$ ).

$$z_n = z_k \quad k > N.$$

Therefore,  $k-N=P$  – length of the sequence period. Then we generate two sequences:

$$z_0, z_1, z_2, z_3, \dots \tag{1.11}$$

$$z_p, z_{p+1}, z_{p+2}, z_{p+3}, \dots \tag{1.12}$$

and compare the elements  $z_i$  and  $z_{p+i}$ ,  $i=1, 2, \dots$  until they coincide, i.e.

$$z_i^* = z_{p+i}^*$$

The number of the coinciding member in the sequence (1.12) determines the length of the aperiodicity segment:

$$L = P + i^*.$$

We formalize the above mentioned in the form of the following algorithm:

Step 1. Accept  $i=1$ ,  $R=0$ , where  $R$  - a feature that the length of the sequence period has not yet been found.

Step 2. Get the next implementation  $z_i$  of the simulated sequence.

Step 3. Set  $i=i+1$ .

Step 4. Check the condition  $R = 0$ . If it is violated, go to step 7.

Step 5. Verification of the fulfillment of the condition  $i=N+1$ , in violation of which go to step 2.

Step 6. Accept  $Y=z_N$  and  $R=R+1$ . Return to step 2.

Step 7. Verification of the condition  $z_i=Y$ . If it is violated, return to step 2.

Step 8. Define  $P=i-N$  and accept  $i=1$ .

Step 9. Re-get the implementations  $z_i$  and  $z_{p+i}$  of the sequence being studied.

Step 10. Check the condition  $z_i = z_{p+i}$  and, if it is met, go to step 12.

Step 11. Set  $i=i+1$ . Return to step 9.

Step 12. Define  $L=P+i$ .

Step 13. Obtain  $P$  and  $L$ .

## 1.6 Perturbation method

For modeling sequences of uniformly distributed random numbers with super-large values of  $L$  and  $P$  Golenko DI proposed a perturbation method [35]. The idea of this method is that recurrence relations are used to simulate sequence numbers of the following form:

$$z_{j+1} = \begin{cases} \Phi(z_j), & \text{if } j \neq M, \\ \Psi(z_j), & \text{if } j = M. \end{cases} \quad (1.13)$$

The parameter  $M$  is selected from the condition  $M < L$  and is called the perturbation period. From the expression (1.13) it can be seen that using the function  $\Phi(z)$  the sequence  $z_0, z_1, z_2, \dots, z_{M-1}$  is modeled.

Then, in order to avoid the possible coincidence  $z_L=z_1$  ( $l < L$ ), there is generated a random number  $z_M$  using the function  $\Psi(z)$ . Such a perturbation of the sequence is repeated every  $M-1$  random numbers until the specified length of the sequence is reached. The algorithm that implements the perturbation method consists of seven steps.

- Step 1. Set  $k=1$  and  $j=0$ .
  - Step 2. Check the condition  $j=k*M$ . If this condition is met, go to step 4.
  - Step 3. Implement the recurrence relation  $z_{j+1}=\Phi(z_j)$ . Go to step 6.
  - Step 4. Implement the recurrence relation  $z_{j+1}=\Psi(z_j)$  and set  $k=k+1$ .
  - Step 5. Set  $j=j+1$ .
  - Step 6. Check the condition  $j>N$ , where  $N$  is the finite length of the simulated sequence. If this condition is violated, go to step 2.
  - Step 7. Display the simulation results.
- Golenko D.I. showed that the method he proposed increased the length of the aperiodicity segment of the simulated sequence by  $\sqrt{M}$  times.

### Control questions to the section

1. What conditions must be satisfied in order the period ( $P$ ) of a sequence of random numbers obtained by the formula  $z_{n+1}^* = az_n^* + C(\text{mod } m)$  was equal to  $P=m$ ?
2. What condition determine the parameters of the formula  $z_{n+1}^* = az_n^* (\text{mod } m)$ ?
3. For modeling of which pattern there is used the perturbation method of D.I. Golenko?
4. To which interval do the random numbers obtained by the formula  $z_{n+1} = 10^{-2k} [10^{2k} D[10^k \cdot z_n^2]]$  belong?
5. What method does the formula relate to  $z_{j+1} = z_j + z_{j-k} (\text{mod } m)$ ?
6. Give examples of recurrence relations of the first and second orders for modeling the base sequence.
7. Get the formula for modeling a sequence of random numbers providing the length of the period  $P = 512$ .
8. Define  $P$  and  $L$  sequences:  
0.17; 0.31; 0.02; 0.94; 0.45; 0.27; 0.13; 0.78; 0.22; 0.45; 0.27.
9. Let  $z_0 = 0,221$ . Define  $z_1$  by truncation method.
10. What method is the sequence simulated? 0.15; 0.97; 0.45; 0.36; 0.62.

## Chapter 2. Random events modeling

### 2.1 Simple event modeling

In section 1.1, there was noted that the basic principle of modeling various random patterns is reduced to modeling the basic sequence of random numbers with a uniform distribution on the interval  $[0, 1]$  and to its subsequent transformation.

For the convenience of transformations, we represent random numbers of a uniform sequence in the form of independent implementations  $z$  of a random variable  $\xi$ . The designation  $\xi$  is hereinafter assigned only to a random variable having a uniform distribution on the interval  $[0, 1]$ .

The use of random numbers for modeling random patterns will begin with the simplest pattern - a random event.

Let an event  $A$  occurs with probability  $P(A)=p$ .

We formulate the theorem 2.1 [29]. Let  $p$  - a probability of occurrence of the event  $A$ , and  $z$  - an independent implementation of the random variable  $\xi$ . Then, in order the event  $A$  occur, it is necessary to fulfill the conditions  $z \leq p$ .

$$\text{Proof. } p = P(A) = P\{\xi \leq p\} = \int_0^p f(z)dz = \int_0^p 1dz = p.$$

An algorithm that implements the condition of this theorem includes 7 steps.

Step 1. Set  $j=1$ .

Step 2. Obtain an implementation  $z$  of a random variable  $\xi$ .

Step 3. Check the condition  $z \leq p$ . If this condition is violated, go to step 5.

Step 4. Fix the number  $S$  of the occurrence of an event  $A$ .

Step 5. Set  $j=j+1$ .

Step 6. Check condition  $j > n$ , where  $n$  - number of independent tests.

If this condition is violated, go to step 2.

Step 7. Display the value of  $S$ .

### 2.2 Modeling a complete group of events

Let  $A_1, A_2, \dots, A_n$  - complete group of incompatible events with given probabilities:

$$\left( \begin{array}{c} A_1, A_2 \dots A_n \\ p_1, p_2 \dots p_n \end{array} \right),$$

where  $p_k = P\{A_k\}$ ,  $k = 1, 2, \dots, n$ ,  $\sum_{k=1}^n p_k = 1$ .

For modeling the complete group of events, we also use random numbers  $z \in [0, 1]$ . First, we divide the segment  $[0, 1]$  into sections  $\Delta_k$  such that the length  $\Delta_k = p_k$  (Figure 2.1.).

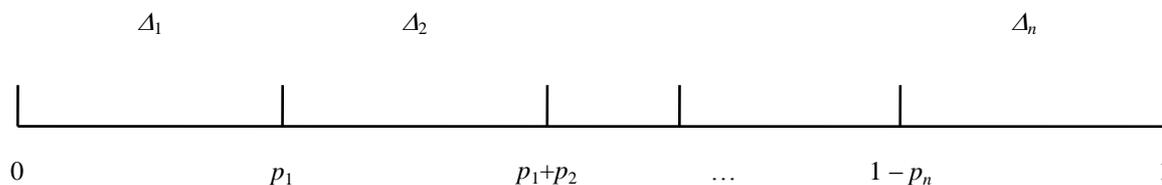


Figure 2.1. - A segment  $[0, 1]$ , divided into sections  $\Delta_k$

Formulate theorem 2.2. Let  $z$  - an independent implementation of the random variable  $\xi$ . Then each of the incompatible events  $A_k$ , forming a complete group of events, occurs with probability  $p_k$  when the condition  $z \in \Delta_k$  is fulfilled.

Proof [29].

$$P(A_k) = P\{z \in \Delta_k\} = \Delta_k = p_k.$$

An algorithm modeling a complete group of events and based on theorem 2.2 implements the following steps.

Step 1. Set  $j=1$ .

Step 2. Obtain the implementation  $z$  of the random variable  $\xi$ .

Step 3. Set  $k=1$ .

Step 4. Check the condition  $z \in \Delta_k$ . If the condition is met, go to step 6.

Step 5. Set  $k=k+1$ . Return to step 4.

Step 6. Fix the results of the occurrence of the  $A_k$ . ( $S_k=S_k+1$ ).

Step 7. Set  $j=j+1$ .

Step 8. Check the condition  $j>n$ , where  $n$  - the given number of experiments.

If this condition is violated, go to step 2.

Step 9. Display the values of  $\{S_k\}$ .

The verification of the condition  $z \in \Delta_k$  in step 4 is usually implemented as follows. The random number  $z$  obtained in step 2 is compared with  $p_1$ . If  $z < p_1$ , then an event  $A_1$  occurred, if  $z > p_1$ ,  $z$  is compared with  $p_1+p_2$ . If  $z < p_1+p_2$ , then an event  $A_2$  occurred, otherwise  $z$  is compared with  $p_1 + p_2 + p_3$ , etc.

The average number of such comparisons is determined by the formula [29]:

$$t = \sum_{k=1}^{n-1} k \cdot p_k + (n-1)p_n.$$

A number of methods for decreasing the value of  $t$  are known, from which a simple ordering method can be distinguished, consisting in the fact that the  $A_k$  events are renumbered so that the values of  $p_k$  are arranged in decreasing order.

## 2.3 Complex events modeling

An event is complex if the result of its outcome depends on two or more events. Complex events are divided into independent and dependent. A complex event is independent if its simple events are also independent. For example, two students living in the same dorm room must pass an exam. Let event  $A$  correspond to the successful passing of the exam by the first student, and event  $B$  – by the second one. Obviously, these simple events can be considered independent. Possible outcomes of a complex event in this case are events  $AB, A\bar{B}, \bar{A}B, \bar{A}\bar{B}$  with probabilities:

$$p_a p_b, p_a(1-p_b), (1-p_a)p_b, (1-p_a)(1-p_b).$$

For modeling joint experiments, we use the conditions of theorem 2.1.

An algorithm that implements the conditions of this theorem applied to a complex event consisting of two simple events  $A$  and  $B$  has the following form [36]:

Step 1. Set  $j=1$ .

Step 2. Obtain two independent implementations  $z_j, z_{j+1}$  of the random variable  $\xi$ .

Step 3. Verify that the conditions are met.

$$z_j \leq p_a \text{ and } z_{j+1} \leq p_b.$$

Step 4. Depending on the results of comparisons in step 3, add one to one of the counters:

$$S_{AB} = S_{AB} + 1, \quad S_{\bar{A}B} = S_{\bar{A}B} + 1, \quad S_{A\bar{B}} = S_{A\bar{B}} + 1, \quad S_{\bar{A}\bar{B}} = S_{\bar{A}\bar{B}} + 1.$$

Step 5. Set  $j=j+2$ .

Step 6. Verify the condition  $j < 2n$ , where  $n$  - the given number of experiments.

If this condition is met, go to step 2.

Step 7. Display the values of counters.

Let consider the case when events  $A$  and  $B$  are dependent. In the previous example about two students, we will assume that they are newlyweds and the possible failure of one of them will significantly affect the budget of the young family and will have a certain impact on the outcome of the exam for the second student. In this case, except the probabilities  $p_a$  and  $p_b$ , there is given the conditional probability  $p_{B/a}$ .

During the modeling complex dependent events, a simple event modeling scheme based on theorem 2.1 is also convenient.

A specific algorithm for modeling complex dependent events, consisting of two simple events, is not much more complicated than the previous algorithm.

Step 1. Set  $j=1$ .

Step 2. Calculate the conditional probability  $P(B/\bar{A}) = P_{B/\bar{A}}$ .

Step 3. Obtain the implementation  $z_j$  of the random variable  $\xi$ .

Step 4. Verify the condition  $z_j < p_a$  is met.

The result of this verification is recorded by one of the following two counters:

$$S_A = S_A + 1, S_{\bar{A}} = S_{\bar{A}} + 1.$$

Step 5. Obtain the implementation  $z_{j+1}$  of the random variable  $\xi$ .

Step 6. Depending on the comparison result, in step 4 verify that one of the following two conditions is met:

$$z_{j+1} \leq P_{B/A} \text{ or } z_{j+1} \leq P_{B/\bar{A}}.$$

Step 7. According to the results of comparisons in step 6, add one to one of the counters:

$$S_{AB} = S_{AB} + 1, S_{\bar{A}B} = S_{\bar{A}B} + 1, S_{A\bar{B}} = S_{A\bar{B}} + 1, S_{\bar{A}\bar{B}} = S_{\bar{A}\bar{B}} + 1.$$

Step 8. Set  $j=j+2$ .

Step 9. Verify the condition  $j < 2n$ .

If this condition is met, go to step 3.

Step 10. Display the values of counters.

In order to calculate the conditional probability  $P_{B/\bar{A}}$  in step 2, there is used the total probability formula:

$$P_{B/\bar{A}} = \frac{P_B - P_A P_{B/A}}{1 - P_A}.$$

### Control questions to the section

1. Define the event according to a given condition:  $\left| \begin{array}{ccc} A_1 & A_2 & A_n \\ 0.2 & 0.5 & 0.3 \end{array} \right|$  at  $z=0.14$ .

2. Obtain one implementation of the complete event group:  $\left( \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0.2 & 0.2 & 0.3 & 0.1 & 0.2 \end{array} \right)$  at  $z=0.714$ .

3. Let a complex event consist of simple events  $A$  and  $B$ . What initial data should be known for modeling independent complex events?

4. Let a complex event consist of simple events  $A$  and  $B$ . Define 2 implementations of a complex dependent event at:

$$p_A = 0.4; \quad p_B = 0.5; \quad p_{B/A} = 0.6;$$

$$z_1 = 0.7; \quad z_2 = 0.5; \quad z_3 = 0.25; \quad z_4 = 0.9$$

5. How can a table from the first control question be converted in order to increase the efficiency of the modeling algorithm?

## Chapter 3. Continuous random values modeling

### 3.1 Classification of continuous random variables modeling methods

For random variables modeling with a given distribution law, as it follows from the basic principle of random objects modeling, it is necessary to transform the base random variable  $\xi$ . Four directions of such a transformation can be distinguished: analytical, selective, probabilistic, and combined.

In the analytical transformation over the implementation  $z$  of a random variable  $\xi$ , some operation is performed that forms the number  $x$ , which can be considered as the implementation of a random variable  $\eta$  with a given distribution law. The most famous is the inverse function method.

However, for a number of important distributions for which the distribution law is not expressed in terms of elementary functions, this method is almost impossible to implement.

The essence of the next direction is to select numbers from the base sequence so that they form a new sequence with the desired distribution.

Widespread among the "selective" methods is the method of exclusion of Neumann. Unfortunately, this method is characterized by a large number of "idle strokes" and is used to model random variables that are defined on a finite interval  $[a, b]$ .

The third direction is related to modeling the conditions of the corresponding limit theorems of probability theory, providing an almost acceptable approximation to the given distribution laws. Obviously, the scope of this direction is limited by the number of limit theorems.

The name of the fourth direction predetermines the use of several methods simultaneously for modeling random variables. This approach is justified in modeling continuous random variables with rather complicated distribution laws.

Below we will consider one method from each direction. In our opinion, these methods, complementing each other, will provide continuous random variables modeling with any distribution law, specified both analytically and in the form of graphs or tables.

### 3.2 Inverse function method

Let a continuous random variable  $\eta$  is defined in the interval  $(a,b)$  and has a distribution density  $f(x)>0$  at  $a<x<b$  (case  $a=-\infty$ ,  $b=\infty$  is not excluded).

$$\text{Distribution function } F(x) = \int_a^x f(x)dx.$$

We formulate the principle of the inverse function method in the form of theorem 3.1 [29]. A random variable  $\eta$  whose  $x$  implementations are determined from the expression:

$$F(x) = z \text{ OR } x = F^{-1}(z), \quad (3.1)$$

where  $z$  – implementation of a basic random variable  $\xi$ , has a distribution density  $f(x)$ .

*Proof.*

Let write an expression for the probability of setting a random variable  $\xi$  in an interval  $[0, z]$ :

$$P\{\xi \leq z\} = P\{F(\eta) \leq F(x)\} = P\{\eta \leq x\} = F(x) = z. \quad (3.2)$$

The first equality of expression (3.2) is written from condition (3.1) of this theorem. The validity of the second equality follows from the property of a monotonic increase in the distribution function from zero to unity. And finally, the last equality is predetermined by the well-known property of a uniform distribution so that the probability of a random variable setting into a certain interval is equal to the length of this interval, i.e.  $P\{\xi < z\} = z$ .

For the practical application of the inverse function method, it is necessary to solve the equation for  $x_j$

$$\int_a^{x_j} f(x) dx = z_j. \quad (3.3)$$

*Example 3.1.* A random variable  $\eta$  with density function  $f(x) = x^{-2}$  is defined in the interval  $[1, \infty)$ . Using relation (3.3), we can obtain:

$$\int_1^{x_j} x^{-2} dx = 1 - \frac{1}{x_j} = z_j,$$

then:

$$x_j = F^{-1}(z_j) = \frac{1}{1 - z_j}.$$

An algorithm that implements the inverse function method consists of the following procedures:

Step 1. Set  $j=1$ .

Step 2. Obtain the implementation  $z_j$  of the random variable  $\xi$ .

Step 3. Calculate the implementation  $x_j$  of the random variable  $\eta$ .

$$x_j = F^{-1}(z_j).$$

Step 4. Set  $j=j+1$ .

Step 5. Verify that  $j > n$  is satisfied, where  $n$  – the required number of implementations of the random variable  $\eta$ . If this condition is violated, go to step 2.

Step 6. Display the values of  $\{x_j\}$ .

### 3.3 Exclusion method by Neumann

The essence of the exclusion method proposed by John von Neumann is that some of the numbers are excluded from the evenly distributed basic sequence so that the remaining numbers obey the given law.

Let the random variable  $\eta$  be defined on the interval  $[a, b]$  and have a density function  $f(x) < M$  bounded above (Figure 3.1):

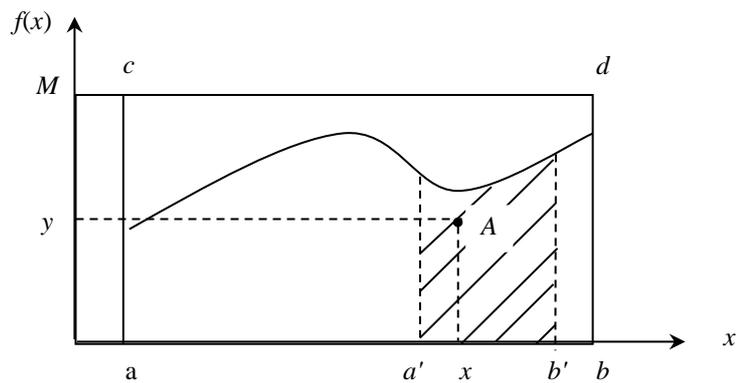


Figure 3.1 - Exclusion method by Neumann

Let formulate the theorem 3.2. Let  $z_1$  and  $z_2$  – independent implementations of the basic random variable  $\xi$  and:

$$x = a + z_1(b - a), \quad y = Mz_2. \quad (3.4)$$

Then random variable  $\eta$ , defined from the condition:

$$\eta = x \text{ and } y < f(x), \quad (3.5)$$

has the distribution density  $f(x)$ .

Proof [29]. Random points  $A(x, y)$ , whose coordinates are found by formulas (3.4), have a uniform distribution in the rectangle  $abcd$ , whose area is  $M(b-a)$ . Find the probability that the point  $A(x, y)$  is under the curve  $y=f(x)$ :

$$P\{y < f(x)\} = \int_a^b f(x)dx [M(b-a)]^{-1} = [M(b-a)]^{-1}.$$

The probability that the point  $A(x, y)$  is lower than the curve  $y=f(x)$  on the interval  $[a', b']$  is also equal to the ratio of the areas:

$$P\{a' \leq x \leq b', y < f(x)\} = \frac{\int_{a'}^{b'} f(x) dx}{M(b-a)}.$$

Calculate the conditional probability:

$$P\{a' \leq x \leq b' / y < f(x)\} = \frac{P\{a' \leq x \leq b', y < f(x)\}}{P\{y < f(x)\}} = \frac{\int_{a'}^{b'} f(x) dx}{M(b-a)},$$

that should be proven.

For the practical implementation of the exclusion method, the following algorithm can be proposed:

Step 1. Set  $i=1, j=1$ .

Step 2. Obtain two independent implementations  $z_{2j-1}$  and  $z_{2j}$  of the random variable  $\xi$ .

Step 3. Calculate  $x_j = a + z_{2j-1}(b-a)$  and  $y_j = M z_{2j}$ .

Step 4. Verify the condition  $y_j < f(x_j)$ . If it is violated, go to step 6.

Step 5. Set  $x_i = x_j$  and  $i = i + 1$ .

Step 6. Set  $j = j + 1$ .

Step 7. Verify the condition for ending the count  $i > n$ .

If this condition is violated, go to step 2.

Step 8. Display of  $\{x_i\}$ .

The effectiveness of the Neumann exclusion method is directly proportional to the probability that the point  $A(x, y)$  falls under the curve  $y = f(x)$ , i.e.

$$P\{y < f(x)\} = [M(b-a)]^{-1}.$$

Therefore, the efficiency of the method will be greatest if we select the smallest possible value of  $M$ , i.e. if we accept:

$$M = \text{SUP}_{a \leq x \leq b} f(x).$$

*Example 3.2.*  $f(x) = 2x + x^2, x \in [1, 5]$ .

*Solution.* Let define the parameter  $M$ :

$$M = \text{SUP} f(x) = \text{SUP}_{1 \leq x \leq 5} (2x + x^2) = 35.$$

Suppose that in step 2 we obtained  $z_1 = 0.75, z_2 = 0.2$ .

Then  $x_1 = 1 + 0.75 * 4 = 4;$

$y_1 = 35 * 0.2 = 7;$

$$f(x_1)=2*4+16=24;$$

$$y_1=7<f(x_1)=24.$$

Therefore,  $\eta=x_1=4$ .

An important advantage of the exclusion method by Neumann is the ability to set the distribution law of a random variable both analytically and graphically.

### 3.4 Limit theorems method

This method of random variables modeling is based on an approximate reproduction of the conditions under which the corresponding limit theorems turn out to be valid. Thus, the central limit theorem of probability theory allows to obtain a random variable with a normal distribution law. This theorem was firstly formulated by Laplace. Many outstanding mathematicians were involved in its generalization, including Chebyshev P.L., Markov A.A. and Lyapunov A.M.

Let give the central limit theorem in the following statement:

*Theorem 3.3.* Let  $\tau_1, \tau_2, \dots, \tau_n$  – mutually independent normalized random variables with the same distribution. Then, for  $n \rightarrow \infty$  the distribution of the normalized sum:

$$\eta^H = \frac{1}{\sqrt{n}} \sum_{i=1}^n \tau_i, \quad (3.6)$$

tends to normal distribution with probability density:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

A proof of this theorem can be found in textbooks on probability theory, for example [6].

If we use the basic random variables  $\xi_i$  with mathematical expectation  $m_x = 1/2$  and the dispersion  $\sigma_x^2 = 1/12$  as independent random variables  $\tau_i$ , then the normalized sum (3.6) takes the form:

$$\eta^H = \sqrt{\frac{3}{n}} \sum_{i=1}^n (2\xi_i - 1). \quad (3.7)$$

Therefore, according to the formula (3.7) for sufficiently large  $n$ , we can calculate the approximate values of the normal distribution with parameters  $m=0$  and  $\sigma^2=1$ . Studies have shown that even at  $n=12$ , the error in the sum (3.7) does not exceed  $9*10^{-3}$ . Therefore, in practice, for normal distribution law modeling with given parameters  $m_x$  and  $\sigma_x^2$ , there is the formula:

$$x = m_x + \sigma_x \sum_{i=1}^{12} (x_i - 6). \quad (3.8)$$

Here  $z$  and  $x$  - implementations of, respectively, basic ( $\xi$ ) and simulated ( $\eta$ ) random variables.

The algorithm implementing the formula (3.8) is as follows:

Step 1. Set  $j=1$ .

Step 2. Accept  $S=0$  and  $i=1$ .

Step 3. Obtain the implementation  $z$  of the random variable  $\xi$ .

Step 4. Set  $S=S+z$  and  $i=i+1$ .

Step 5. Verify the condition  $i \leq 12$ . If the condition is met, go to step 3.

Step 6. Calculation of the next implementation  $x_j$  of the random variable  $\eta$ :

$$x_j = m_x + \sigma_x (S-6).$$

Step 7. Set  $j=j+1$ .

Step 8. Verify the count ending condition  $j > n$ , where  $n$  - the number of normal distribution implementations. If this condition is violated, return to step 2.

Step 9. Display the implementation of  $\{x_j\}$ .

### 3.5 Composition method

If the distribution function  $F(x)$  of some random variable  $\eta$  has a rather complicated form, then in many cases it can be represented as a composition of simpler distributions:

$$F(x) = \sum_{k=1}^m C_k \cdot F_k(x) \quad (3.9)$$

where  $C_k > 0$ . From (3.9) at  $x \rightarrow \infty$  it follows that  $\sum_{k=1}^m C_k = 1$ . Therefore, we can enter the complete group of events  $\{A_k\}$  by a given table:

$$\begin{pmatrix} A_1 A_2 \dots A_m \\ C_1 C_2 \dots C_m \end{pmatrix},$$

where  $C_k = P(A_k)$ .

*Theorem 3.4.* [22] Let  $\xi_1$  and  $\xi_2$  – independent basic random variables. If, using  $\xi_1$ , we define the number of the occurred event  $A_k$ , and then from the equation  $F_k(\eta) = \xi_2$  we obtain  $\eta$ , then the distribution function  $\eta$  is equal to  $F(x)$ .

*Proof.* Let use the total probability theorem and calculate the distribution

function of the random variable  $\eta$ :

$$F(x) = P\{\eta < x\} = \sum_{k=1}^m P\{\eta < x / A_k\}P(A_k) = \sum_{k=1}^m F_k(x)C_k = F(x),$$

that should be proven.

For the practical implementation of the composition method, it is more convenient to use the distribution density function  $f(x)$  of the simulated random variable  $\eta$ . Indeed, in this case, the coefficients  $C_k$  of the sum:

$$f(x) = \sum_{k=1}^m C_k \cdot f_k(x) \quad (3.10)$$

are easily defined as parts of a unit area under the function  $f(x)$  (Figure 3.2).

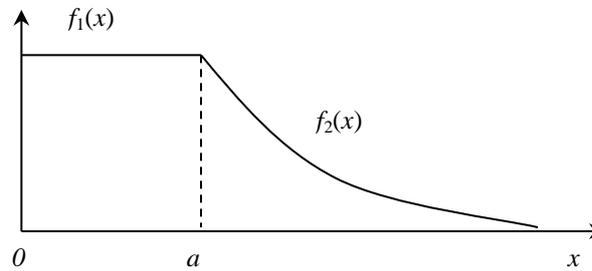


Figure 3.2 - Area of the  $f(x)$ function

An algorithm that implements the conditions of theorem 3.4 consists of seven steps [35]:

Step 1. Set  $j=1$ .

Step 2. Obtain the implementations  $z_{2j-1}$  and  $z_{2j}$  of the random variable  $x$ .

Step 3. Using  $z_{2j-1}$  play an event  $A_k$ .

Step 4. Obtain the implementation  $x_j$ , corresponding to the density function  $f_k(x)$ .

Step 5. Set  $j=j+1$ .

Step 6. Verify that  $j>n$  is met, where  $n$  – the required volume of implementations of the random variable  $\eta$ . If this condition is violated, return to step 2.

Step 7. Display the results of implementations.

### 3.6 Special distribution modeling

Let consider modeling methods for the following most common continuous distributions: normal, uniform, exponential, linear and gamma distribution.

#### 3.6.1 Normal distribution

A normal or Gaussian distribution is one of the most important and frequently used continuous distributions.

Normal distribution density function:

$$f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{(x-m_x)^2}{2\sigma_x^2}}, \quad -\infty < x < \infty.$$

Mathematical expectation is  $M[\eta] = m_x$ .

Dispersion is  $D[\eta] = \sigma_x^2$ .

For modeling of a normally distributed random variable  $\eta$ , there are several algorithms. One of them, based on the method of limit theorems, is considered in Section 3.4. Another method was proposed by the Americans Box G. and Müller M. [12, 32] and is called the polar coordinate method.

Let  $\eta$  and  $\tau$  – two independent random variables with normalized normal distribution laws. Their implementations are denoted by  $x$  and  $y$ , respectively. Represent  $\eta$  and  $\tau$  by the coordinates of the point  $A$  in the Cartesian coordinate system. Then the polar coordinates of the point  $A$  are the quantities:

$$R = \sqrt{\eta^2 + \tau^2} \quad \text{and} \quad \theta = \text{arctg} \frac{\tau}{\eta},$$

with density functions:

$$f(r) = re^{-\frac{r^2}{2}} \quad \text{and} \quad f(\varphi) = \frac{1}{2\pi},$$

where  $r$  and  $\varphi$  – implementations of random variables  $R$  and  $\theta$ .

Then, for example, by calculating, by the inverse implementation function of  $r$  and  $\varphi$  polar coordinates, we can find the implementations of  $x$ ,  $y$  Cartesian coordinates of the point  $A$ :

$$\begin{aligned} x &= r \cos \varphi = \sqrt{-2 \ln z_1} \cos 2\pi z_2, \\ y &= r \sin \varphi = \sqrt{-2 \ln z_1} \sin 2\pi z_2. \end{aligned} \quad (3.11)$$

Obviously, for random variable implementation modeling with a normal distribution law, any of formulas (3.11) can be used. For example, the first of them:

$$x_j = m_x + \sigma_x \sqrt{-2 \ln z_{2j-1}} \cos 2\pi \cdot z_{2j}.$$

Taking into account that calculations by formulas (3.11) are associated with significant costs of computer time, the American Marsalia G. [32] modified the polar coordinate method.

Marsalia Algorithm:

Step 1. Set  $j=1$ ,  $i=1$ .

Step 2. Obtain two independent implementations  $z_{2j-1}$  and  $z_{2j}$  of the basic random variable  $\xi$ .

Step 3. Accept  $V_1=2z_{2j-1}$ ,  $V_2=2z_{2j}$ ,  $S=V_1^2 + V_2^2$  and  $j=j+1$ .

Step 4. Verifying the condition  $S \leq 1$ . If this condition is violated, return to step 2.

Step 5. Calculating implementations  $x_i$  and  $y_i$  using the formulas:

$$x_i = m_x + \sigma_x V_1 \sqrt{\frac{-2 \ln S}{S}} \quad \text{and} \quad y_i = m_x + \sigma_x V_2 \sqrt{\frac{-2 \ln S}{S}}.$$

Accept  $i=i+1$ .

Step 7. Verifying the condition  $i > n$ , where  $n$  – required number of implementations. If this condition is violated, go to step 2.

Step 8. Display  $x_i$  and  $y_i$ .

For one random variable implementation modeling in step 5, it is sufficient to use only one of the formulas (3.11).

### 3.6.2 Logarithmically - normal distribution

A random variable is called logarithmically - normal, or lognormal if its natural logarithm obeys the normal distribution law.

The logarithmic transformation of a random variable is used to approximate the asymmetric distribution to the normal law [6]. The asymmetric distribution of random variables is characteristic to the phenomena described by positive values. The lognormal distribution is used, for example, in modeling variables such as biological life expectancy, time of failure of tools, income distribution, etc.

So, if the random variable  $x$  has a lognormal distribution, and the quantity  $y = \ln x$ , then the density function  $y$  has the following form:

$$f(y) = \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{(y-m_y)^2}{2\sigma_y^2}},$$

$m_y$  and  $\sigma_y^2$  – respectively, the mathematical expectation and dispersion of  $\ln x$ .

Then the distribution function of the lognormal law has the following form:

$$F(y) = \frac{1}{\sigma_y \sqrt{2\pi}} \int_0^y \exp\left[-\frac{(y-m_y)^2}{2\sigma_y^2}\right] dy.$$

The algorithm for modeling this law consists of the following steps:

The initial step. Let the mathematical expectation  $m_y$  and dispersion  $\sigma_y$  of a random variable  $\ln x$  are given.

Step 1. Modeling the normalized implementation of  $r$  of the normal distribution law with density function:

$$f(r) = \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}}.$$

Step 2. Calculate  $y$  from expression  $y = m_y + r\sigma_y$ .

Step 3. Obtain the implementation of the lognormal distribution law  $x$  according to  $x = e^y$ .

Step 4. Display the value of  $x$ .

### 3.6.3 Uniform distribution

The uniform distribution in frequency of application is second only to the normal law and is characterized by a density function:

$$f(x) = \frac{1}{b-a}, \quad x \in [a, b].$$

Mathematical expectation:

$$M[\eta] = \frac{a+b}{2}.$$

Dispersion:

$$D[\eta] = \frac{(b-a)^2}{12}.$$

For the random variable  $\eta$  modeling with a uniform distribution, we use the formula:

$$x = a + z(b-a), \quad (3.13)$$

obtained by the inverse function method.

Algorithm:

Step 1. Set  $j=1$ .

Step 2. Obtain the implementation  $z$  of the basic random variable  $\eta$ .

Step 3. Calculate  $x_j = a + z_j(b-a)$ .

Step 4. Accept  $j=j+1$ .

Step 5. Verify the condition  $j > n$ , where  $n$  – required volume of implementations of a random variable  $\eta$ . If this condition is violated, return to step 2.

Step 6. Display the obtained implementations.

### 3.6.4 Exponential distribution

The exponential distribution describes a number of real processes where the "appearance time" is considered. For example, the uptime of electronic devices, the time intervals between phone calls or between major earthquakes, etc.

Exponential distribution density function:

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0 \quad .$$

Mathematical expectation  $M[\eta] = m_x = \frac{1}{\lambda}$ .

Dispersion  $D[\eta] = \sigma_x^2 = \frac{1}{\lambda^2}$ .

To simulate this distribution, we use the inverse function method:

$$F(x) = \lambda \int_0^{x_j} e^{-\lambda x} dx = 1 - e^{-\lambda x} \Big|_0^{x_j} = z_j.$$

Therefore, the inverse will be the function:

$$x_j = -\frac{1}{\lambda} \ln(1 - z_j). \quad (3.14)$$

Considering that the random variable  $\xi_j = 1 - \xi$  also has a uniform distribution on the interval  $[0, 1]$ , the relation (3.14) can be replaced by the formula:

$$x_j = -\frac{1}{\lambda} \ln z_j. \quad (3.15)$$

Algorithm:

Step 1. Set  $j=1$ .

Step 2. Obtain the implementation  $z$  of the random variable  $\xi$ .

Step 3. Calculate the implementation  $x_j$  by the formula (3.15) of the random variable  $\eta$ . Set  $j=j+1$ .

Step 4. Verify the count ending condition  $j > n$ . If this condition is violated, return to step 2.

Step 5. Display  $\{x_j\}$ .

### 3.6.5 Linear distribution

The distribution density of the linear law is expressed by the formula:

$$f(x) = \lambda \left(1 - \frac{\lambda}{2} x\right), \quad x \in \left(0, \frac{2}{\lambda}\right] \quad .$$

3.3. A graph of the distribution density of this random variable is shown on Figure

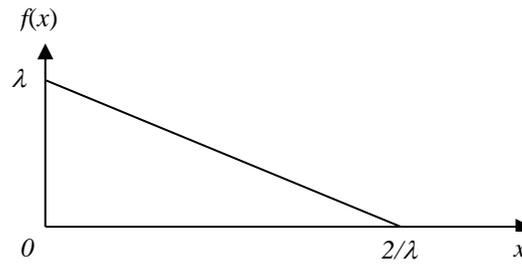


Figure 3.3 - Linear distribution density function

Mathematical expectation  $M[\eta] = m_x = \frac{2}{3\lambda}$ .

Dispersion  $D[\eta] = \sigma_x^2 = \frac{2}{9\lambda^2}$ .

The formula for calculating the implementation of the linear distribution can be obtained by the inverse function method:

$$F(x) = \lambda \int_0^{x_j} \left(1 - \frac{\lambda}{2}x\right) dx = \lambda \left(x_j - \frac{\lambda}{4}x_j^2\right) = z_j.$$

Then,

$$x_j = \frac{2}{\lambda} \left(1 - \sqrt{z_j}\right). \quad (3.16)$$

Algorithm:

Step 1. Set  $j=1$ .

Step 2. Obtain the implementation  $z_j$  of the random variable  $\xi$ .

Step 3. Calculate  $x_j$  by the formula (3.16) and accept  $j=j+1$ .

Step 4. Verify the count ending condition  $j>n$ . If this condition is violated, return to step 2.

Step 5. Display  $\{x_j\}$ .

### 3.6.6 Gamma-distribution

Random variable  $\eta$  with density function:

$$f(x) = \alpha^k [(k-1)!]^{-1} x^{(k-1)} \cdot e^{-\alpha x}, \quad (3.17)$$

where  $\alpha>0, k>0, x\geq 0$  obeys gamma-distribution.

Mathematical expectation and dispersion are respectively equal:

$$M[\eta] = m_x = \frac{k}{\alpha}, \quad D[\eta] = \sigma_x^2 = \frac{k}{\alpha^2}.$$

Many non-negative quantities characterizing any random events can be described using the gamma-distribution. Indeed, since the parameters  $\alpha$  and  $k$  determine the scale and form of the distribution law (Figure 3.4), then due to the change of these parameters values the gamma-distribution density can take many different forms and, therefore, makes this law one of the most universal and valuable in applied relation [34].

In order to determine the values of the gamma-distribution parameters, we can use the expression:

$$K = \frac{m_x^2}{\sigma_x^2} \quad \text{and} \quad \alpha = \frac{m_x}{\sigma_x}.$$

The gamma-distribution is also interesting because it is closely related to a number of other useful distributions. For example, if  $k=1$  and  $\alpha$  – a constant, then a random variable with an exponential distribution is simulated. If  $k$  has an integer value, then the gamma-distribution is usually called the Erlang distribution of the  $k$ -th order. Further, if  $\alpha = 1$ , then with  $k$  increase the gamma distribution approaches the normal level. Setting  $k=l/2$  and  $\alpha = 2$  ( $l$  - the number of degrees of freedom), we can obtain the distribution  $\chi^2$ .

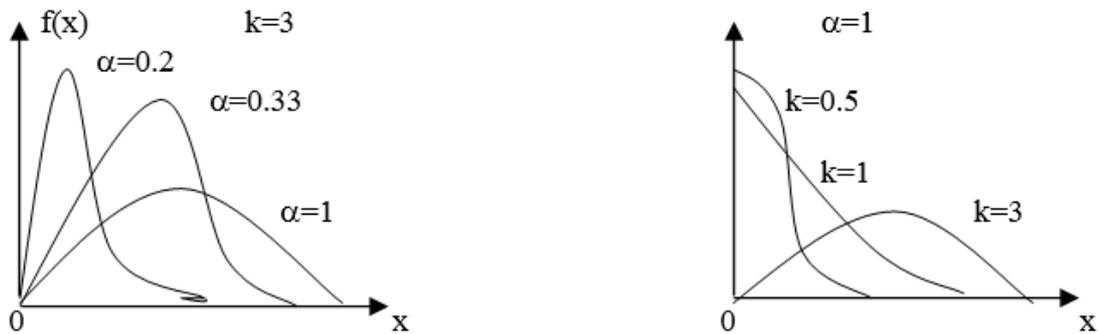


Figure 3.4 - The nature of the influence of the parameters on the form of the density function

The implementation of the gamma-distribution is calculated by the formula:

$$x = -\frac{1}{\alpha} \ln(z_1 \cdot z_2 \cdot \dots \cdot z_k),$$

which is a generalization of the formula  $x = -\frac{1}{\alpha} \ln z$ , obtained by the inverse function method at  $k=1$ .

Algorithm for gamma-distribution modeling:

Step 1. Set  $i=1$ .

Step 2. Set  $S=1, j=1$ .

Step 3. Obtain the implementation  $z_j$  of the basic random variable  $\xi$ .

Step 4. Set  $S=S*z$  and  $j=j+1$ .

Step 5. Verify  $j>k$ . If this condition is violated, return to step 3.

Step 6. Calculate  $(x = -\frac{1}{\alpha} \ln S)$  and accept  $i=i+1$ .

Step 7. Verify the count ending condition  $j>n$ . If this condition is violated, return to step 2.

Step 8. Display  $\{x_i\}$ .

### Control questions to the section

1. For which random pattern modeling the formula  $x = m_x + \sigma_x (\sum_{i=1}^{12} z_i - 6)$  ?

2. For which random pattern modeling the formula  $x = r \cos \varphi = \sqrt{-2 \ln z_1} \cos 2\pi z_2$  is intended, and what advantages it has?

3. For which random pattern modeling the formula  $x_j = -\frac{1}{\lambda} \ln(1 - z_j)$  is intended, and what disadvantages it has?

4. For which random pattern modeling the formula  $x_1 = m_x + \sigma_x V_1 \sqrt{(-2 \ln S / S)}$  is intended, and what connection it has with the formula  $x = r \cos \varphi = \sqrt{-2 \ln z_1} \cos 2\pi z_2$  ?

5. For which random pattern modeling the formula  $x_i = a + z(b - a)$  is intended?

6. For which random pattern modeling the formula  $x_i = -\frac{1}{\alpha} \ln(z_1 \cdot z_2 \cdot \dots \cdot z_k)$  is intended, and by what method is it obtained?

7. For which random pattern modeling is the inverse function method used?

8. For which random pattern modeling is the limit theorems method used?

9. For which random pattern modeling is the J. Neumann exclusion method used?

10. Let  $z = 0,6$ , it is necessary to find an implementation of a random variable obeying the distribution law  $P\{\eta \leq x\} = (2x - 3)/5$

11. Let  $z = 0,6$ , it is necessary to find an implementation of a random variable obeying the distribution law  $F(x) = x - 4$ .

## Chapter 4. Discrete random variables modeling

### 4.1 The main method for modeling discrete random variables

Let a discrete random variable  $v$  has a distribution:

$$v = \begin{pmatrix} x_1 & x_2 \dots x_m \\ p_1 & p_2 \dots p_m \end{pmatrix}, \quad (4.1)$$

where  $p_k = P\{v = x_k\}$ .

Taking into account the identical nature of tasks of both the complete group of incompatible events (2.1) and the discrete random variable  $v$  (4.1), in order to model the latter, we can also use theorem 2.2, where the events  $A_k$  are replaced by the values  $x_k$  of the random variable  $v$ .

Although this method of discrete random variables modeling is universal, it requires a significant amount of computer time. Therefore, for modeling a number of discrete distributions that have important practical applications, more efficient methods have been developed.

### 4.2 Geometric distribution

If an event occurs with a probability  $p$ , then the number  $v$  of independent experiments necessary for the occurrence of an event (or the number of experiments between events) obeys a geometric distribution. Therefore,  $v=1$  with a probability  $p$ ,  $v=2$  with a probability  $(1-p)*p$  and in general case:

$$P\{v = k\} = p(1-p)^{k-1} = p_k. \quad (4.2)$$

The mathematical expectation of the geometric distribution is

$$M\{v\} = m_x = (1-p)/p,$$

and dispersion is

$$D\{v\} = \sigma_x^2 = (1-p)/p^2.$$

For random variables modeling with a geometric distribution, when  $p$  is small enough, then we can use the formula:

$$v = [\ln \xi / \ln(1-p)], \quad (4.3)$$

where  $\nu$  is equal to the smallest integer greater than or equal to the value of the expression enclosed in square brackets.

Let show that this formula produces implementations of a random variable with a geometric distribution [10]:

$$P\{\nu = k\} = P\{k-1 \leq \ln \xi / \ln(1-p) \leq k\} = P\{-(k-1)\ln(1-p) \leq -\ln \xi \leq -k \ln(1-p)\} = P\{(1-p)^k \leq \xi \leq (1-p)^{k-1}\} = (1-p)^{k-1} - (1-p)^k = p(1-p)^{k-1}.$$

The algorithm for geometric distribution modeling can be represented as:

Step 1. Set  $j=1$ .

Step 2. Obtain the implementation  $z$  of the random variable  $\xi$ .

Step 3. Calculate the implementation  $x_j$  of the random variable  $\nu$ :

$$x_j = \lfloor \left[ \frac{\ln z}{\ln(1-p)} \right] \rfloor + 1$$

and accept  $j=j+1$ .

Step 4. Verify the count ending condition  $j > n$ . If this condition is violated, return to step 2.

Step 5. Display  $\{x_j\}$ .

### 4.3 Binominal distribution

The binomial distribution law, formulated by the famous Swiss scientist Daniel Bernoulli (1700-1782), is described by a discrete distribution in which a random event can have only two outcomes, called success and failure.

Let any simple event  $A$  have a probability of success  $p = P(A)$ , and a probability of failure  $q = 1 - p$ .

If the implementations of the random variable  $\nu$  characterizing the occurrence of an event  $A$  at  $n$  independent experiments are equal to  $0, 1, \dots, k, \dots, n$ , and the probabilities of these implementations are determined by the expression:

$$p_k = P\{\nu = k\} = C_n^k p^k q^{n-k} \quad (4.4)$$

then this pattern is called binomial distribution.

As can be seen from the formula (4.4), the binomial distribution is described by two parameters:  $p$  and  $n$ .

The mathematical expectation and dispersion of this distribution are respectively equal to:

$$m = np, \quad D = np(1-p).$$

Since an event  $A$  is simple, then the theorem 2.1 from the second chapter is used for modeling this distribution. Then the binomial distribution modeling algorithm consists of the following steps:

Step 1. Initial values are set:  $n$  and  $p$ .

Step 2. Modeling of  $n$  independent implementations  $z_j$   $j = \overline{1, n}$  of the basic random variable  $\xi$ .

Step 3. In accordance with the theorem 2.1 the condition  $z_j \leq p$ ,  $j = \overline{1, n}$  is verified according to the theorem 2.1.

Step 4. For the next implementation of the binomial distribution, there is taken the value  $\nu = k$ , where  $k$  – condition fulfillment number of the step 3.

This algorithm is used for small values of the parameter.  $n$ .

#### 4.4 Poisson distribution

The Poisson distribution is the distribution of rare events and it characterizes the number of events, each of which can occur at any time. For example, the number of powerful earthquakes per year or some other incidents.

The probability that a random variable  $\nu$  takes some integer value  $k$  is determined by the Poisson formula:

$$P\{\nu = k\} = \frac{\lambda^k}{k!} e^{-\lambda}. \quad (4.5)$$

here  $\lambda$  – average number of events per unit time.

Mathematical expectation and dispersion are respectively equal to:

$$M[\nu] = m_x = \lambda,$$

$$D[\nu] = \sigma_x^2 = \lambda.$$

For random variable modeling according to the Poisson law, let use the Poisson limit theorem.

Theorem 4.1. If an event  $A$  occurs with a probability  $p$  in one experiment, then the probability of the occurrence of  $k$  events with  $n$  independent experiments, at  $n \rightarrow \infty$  and  $p \rightarrow 0$  is determined by the formula (4.5).

The scheme for discrete random variable  $\nu$  modeling by the Poisson theorem can be based on the scheme of the binomial distribution described above. Nevertheless, taking into account the conditions of the Poisson theorem:

$$n \rightarrow \infty \text{ and } p \rightarrow 0,$$

the number of experiments is determined from the expression:

$$n = \lambda / p.$$

According to this scheme, we present the following modeling algorithm:

Step 1. Initial values are set:  $n$  and  $p$ .

Step 2. Calculate the value  $e^{-np}$ .

Step 3. Set the value of the initial implementation of the random variable  $\nu$  equal to  $k = 0$ .

Step 4. Obtain the implementation  $z$  of the random variable  $\xi$ .

Step 5. Obtain the product  $k + 1$  of implementations  $z_j$  of the random variable  $\xi$

$$\prod_{j=1}^{k+1} z_j$$

Step 6. Verify the condition:

$$\prod_{j=1}^{k+1} z_j < e^{-np},$$

and if this condition is violated, steps 5 and 6 are repeated for the next value  $k=k+1$ . When this condition is satisfied, the next implementation of the Poisson distribution  $\nu=k$  is calculated.

### Control questions to the section

1. For which random pattern modeling the formula  $x_i = L[\ln(z)/\ln(1-p)] + 1$  is intended?
2. The probability of passing the test to a strict teacher is  $p$ . What law obeys the value characterizing the number of attempts before passing the test?
3. The probability that the bus that arrives at the bus stop will be your is equal to:  $p=0,4$ . Define, what will be your bus by number, if  $z=0,60$ .
4. How is the parameter  $n$  determined in modeling according to Poisson's law?

## Chapter 5. Multidimensional random variables modeling

### 5.1 Sequential modeling method

Modeling of  $n$  –dimensional random variable (vector):

$$\eta = (\eta_1, \eta_2, \dots, \eta_n), \quad (5.1)$$

set by the joint density function:

$$f_{\eta}(x_1, x_2, \dots, x_n),$$

comes to the sequential development of implementations of scalar quantities  $\eta_i$ .

Firstly, let suppose that all the random variables  $\eta_i$  are independent, then:

$$f_{\eta}(x_1, x_2, \dots, x_n) = f_1(x_1) \cdot f_2(x_2) \cdot \dots \cdot f_n(x_n). \quad (5.2)$$

Therefore, each of the random variables  $\eta_i$  can be modeled independently from each other, for example, for example, by the inverse function method:

$$\eta_i = F^{-1}(\xi_i), \quad i=1, 2, \dots, n.$$

If these random variables are dependent on each other, then:

$$f(x_1, x_2, \dots, x_n) = f_1(x_1) \cdot f_2(x_2 / x_1) \cdot \dots \cdot f_n(x_n / x_1 \dots x_{n-1}). \quad (5.3)$$

For the case of a continuous  $n$ -dimensional vector, the conditional density functions can be determined from the following system of equations:

$$\begin{aligned} f_1(x_1) &= \int \dots \int f_{\eta}(x_1, \dots, x_n) dx_2 \dots dx_n; \\ f_2(x_2 / x_1) &= \int \dots \int f_{\eta}(x_1, \dots, x_n) dx_3 \dots dx_n \cdot [f_1(x_1)]^{-1}; \end{aligned} \quad (5.4)$$

$$f_n(x_n / x_1 \dots x_{n-1}) = f_{\eta}(x_1, \dots, x_n) / [f_1(x_1) \cdot \dots \cdot f_{n-1}(x_{n-1} / x_1 \dots x_{n-2})].$$

Let formulate the theorem 5.1 [29]. Let  $\xi_1, \dots, \xi_n$  - independent basic random variables. The set of random variables  $\eta_1, \dots, \eta_n$  obtained by sequential solving the equations:

$$F_1(\eta_1) = \xi_1,$$

$$F_2(\eta_2 / x_1) = \xi_2,$$

$$F_n(\eta_n / x_1 \dots x_{n-1}) = \xi_n,$$

has a joint distribution density  $f_n(x_1, \dots, x_n)$ .

*Proof.* If the values  $\eta_1 = x_1, \dots, \eta_{i-1} = x_{i-1}$  are fixed, then a random variable  $\eta_i$  with distribution function  $F_i(x_i / x_1, \dots, x_{i-1})$  can be defined from the relation:

$$F_i(\eta / x_1, \dots, x_{i-1}) = z_i.$$

Then the probability of inequality  $x_i < \eta_i < x_i + \Delta x_i$  is equal to:

$$P\{x_i < \eta_i < x_i + \Delta x_i / x_1, \dots, x_{i-1}\} \approx f_i(x_i / x_1, \dots, x_{i-1}) \Delta x_i.$$

Consequently, up to infinitely small values of higher orders, the probability of joint fulfillment of  $n$  inequalities is equal to the following product:

$$\begin{aligned} P\{x_1 < \eta_1 < x_1 + \Delta x_1, \dots, x_n < \eta_n < x_n + \Delta x_n\} &= \\ P\{x_1 < \eta_1 < x_1 + \Delta x_1\} \cdot \dots \cdot P\{x_n < \eta_n < x_n + \Delta x_n / x_1 \dots x_{n-1}\} &\approx \\ f_1(x_1) \Delta x_1 \cdot \dots \cdot f_n(x_n / x_1 \dots x_{n-1}) \Delta x_n &= \\ f_n(x_1, \dots, x_n) \Delta x_1 \dots \Delta x_n. & \end{aligned}$$

The theorem is proved.

*Example 5.1* [29].

Let consider a two-dimensional random variable  $\eta = (\eta_1, \eta_2)$  with density function  $f(x_1, x_2) = 6x_2$ , which can take values in a triangle  $x_1 + x_2 < 1$ ,  $x_1 > 0$ ,  $x_2 > 0$ .

Define  $f_1(x_1)$  and  $f_2(x_2/x_1)$ :

$$\begin{aligned} f_1(x_1) &= 6 \int_0^{1-x_1} x_2 dx_2 = 3(1-x_1)^2 \\ f_2(x_1/x_2) &= \frac{f(x_1, x_2)}{f_1(x_1)} = \frac{6x_2}{3(1-x_1)^2} = 2x_2(1-x_1)^{-2}. \end{aligned}$$

Then from the equations of the inverse function method:

$$F_1(x_1) = 3 \int_0^{x_1} (1-x_1)^2 dx_1 = 1 - (1-x_{1i})^3 = z_{1i},$$

$$F_2(x_1/x_2) = 2 \int_0^{x_{2i}} x_2 (1-x_1)^{-2} dx_2 = \frac{x_{2i}^2}{(1-x_1)^2} = z_{2i},$$

let define:

$$x_{1i} = 1 - \sqrt[3]{z_{1i}},$$

$$x_{2i} = (1 - x_{1i}) \sqrt{z_{2i}}.$$

The algorithm of the sequential modeling method consists of two stages - preliminary and main.

*Preliminary stage*

Step 1. Determination of density functions:

$$f_1(x_1), f_2(x_2/x_1), \dots, f_n(x_n/x_1 \dots x_{n-1}).$$

Step 2. Solution of integral equations

$$\int_a^{x_k} f_k(x_k / x_1 \dots x_{k-1}) dx_k = z_k, \quad k=1, 2, \dots, n,$$

relatively to  $x_k$ , i.e. dependency definition:

$$x_k = F_k^{-1}(z_k, x_1, \dots, x_{k-1}), \quad k=1, 2, \dots, n.$$

*Main stage:*

Step 3. Set  $i=1$ .

Step 4. Set  $k=1$ .

Step 5. Obtain the implementation  $z$  of the basic random variable  $\xi$ .

Step 6. Verify the fulfillment of the condition  $k=1$ . If this condition is violated, go to step 8.

Step 7. Calculation of  $x_1 = F_1^{-1}(z)$ . Go to step 9.

Step 8. Calculation of  $x_k = F_k^{-1}(z_k, x_1, \dots, x_{k-1})$ .

Step 9. Accept  $k=k+1$ .

Step 10. Verify the condition  $k>n$ . If this condition is violated, go to step 5.

Step 11. Display  $x[i] = (x_1, x_2, \dots, x_{k-1})$ .

Step 12. Accept  $i=i+1$ .

Step 13. Verify the count ending condition  $i>N$ . If this condition is violated, go to step 4.

Step 14. End.

The method of sequential modeling, unfortunately, is very cumbersome even for the small size of the simulated vector. More economical in terms of the amount of computation required is the Neumann exclusion method.

## 5.2 Generalization of the Neumann exclusion method for vectors modeling

Let the random vector  $\eta=(\eta_1, \dots, \eta_n)$  is also given by the joint density function bounded above:

$$f_{\eta}(x_1, \dots, x_n) \leq f_m.$$

Let suppose that each of the quantities  $\eta_i$  is defined on the interval  $[a_i, b_i]$ ,  $i=1, n$ . Then the modeling of a random vector implementation can be performed according to the following algorithm:

Step 1. Accept  $j=1$ .

Step 2. Obtain the implementation  $s z_1, \dots, z_n, z_{n+1}$  of the basic random variable  $\xi$ .

Step 3. Calculate the values  $x_i = a_i + z_i(b_i - a_i)$ ,  $i=1, n$ .

Step 4. Verify the fulfillment of the condition:

$$f_m \cdot z_{n+1} \leq f_{\eta}(x_1, \dots, x_n).$$

If this condition is violated, go to step 6.

Step 5. Formation of the next implementation of the random vector:

$$x[j] = (x_1, x_2, \dots, x_n).$$

Go to step 7.

Step 6. Set  $j=j-1$ .

Step 7. Accept  $z_0 = z_{n+1}$ .

Step 8. Set  $j=j+1$ .

Step 9. Verify the condition  $j > N$ . If this condition is violated, return to step 2.

Step 10. Display the results.

The efficiency coefficient is equal to:

$$\Theta = \left[ f_m \prod_{i=1}^n (b_i - a_i) \right]^{-1}.$$

and decreases with increasing of vector size.

### 5.3 Method of moments

The joint distribution law of a system of random variables is a complete and comprehensive characteristic of vector random variables. In practice, however, it is not always possible to create adequate laws for vector random variables distribution. Therefore, most often, for these values there are defined mathematical expectations  $M[\eta_i]=m_i$ ,  $i=1,n$ , dispersions  $D[\eta_i]=\sigma_i^2$ ,  $i=1,n$  and the correlation moments  $R_{ij}=M[(\eta_i-m_i)(\eta_j-m_j)]=r_{ij}$ ,  $i=1, n, j=1,n$ .

The method of moments is intended for random vectors modeling defined by these numerical characteristics.

Let a random vector  $\eta=(\eta_1 \dots \eta_n)$  is defined by a vector of mathematical expectations:

$$m_\eta=(m_1, m_2, \dots, m_n)$$

and correlation matrix:

$$R = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ r_{n1} & r_{n2} & \dots & r_{nn} \end{pmatrix}. \quad (5.5)$$

Then a random vector  $\eta$  can be obtained by linear conversion:

$$\eta = m_\eta + A\tau,$$

where  $\tau$  – normalized vector random variable, the coordinates  $\tau_i$  of which obey the normal distribution law. As a matrix  $A$ , we take a triangular matrix of the following form:

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Then, denoting the implementations of the random variables  $\eta$  and  $\tau$  through  $x$  and  $y$ , respectively, we can write specific formulas for a random vector modeling:

$$\begin{aligned} \eta &= (\eta_1, \dots, \eta_n), \\ x_1 &= m_1 + a_{11}y_1, \\ x_2 &= m_2 + a_{21}y_1 + a_{22}y_2, \end{aligned} \quad (5.6)$$

$$x_n = m_n + a_{n1}y_1 + \dots + a_{nn}y_n.$$

In order to determine the coefficients  $a_{ij}$ , we use the elements of the correlation matrix:

$$r_{ij} = M[(x_i - m_i)(x_j - m_j)] = M[x_i x_j] - m_i m_j = M[(a_{i1}y_1 + \dots + a_{ii}y_i)(a_{j1}y_1 + \dots + a_{jj}y_j)].$$

Taking into account that for normalized uncorrelated random variables:

$$M[y_i y_j] = 0, (i \neq j), M[y_i] = 0 \text{ and } M[y_i^2] = 1,$$

elements of matrix  $A$  can be defined from the relation:

$$r_{ij} = a_{i1}a_{j1} + a_{i2}a_{j2} + \dots + a_{ii}a_{ji}. \quad (5.7)$$

In the practical application of the method of moments, it should not be forgotten that this method ensures that the simulated random vector corresponds only to a given correlation moment and does not always guarantee its adequacy to the distribution law. The exception is the normal distribution, where modeling by the method of moments of normal random vectors is equal to modeling using given multidimensional distributions [11, 23].

The method of moments algorithm consists of two stages.

*Preliminary stage:*

Step 1. Define from the relation (5.7) all the elements  $a_{ij}$  of the matrix  $A$ .

*Main stage:*

Step 2. Accept  $k=1$ .

Step 3. Accept  $i=1$ .

Step 4. Obtain an implementation of the normalized normal random variable

$\tau$ .

Step 5. Calculate using the formula:

$$x_i = m_i + \sum_{j=1}^i a_{ij} \cdot y_j,$$

the implementation  $x_i$  of  $i$ -th component of a random vector  $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ .

Step 6. Set  $i=i+1$ .

Step 7. Verify the condition  $i > n$ . If this condition is violated, return to step 4.

Step 8. Display of the  $k$ -th implementation of a random vector:

$$X_k = (x_1, x_2, \dots, x_n).$$

Step 9. Set  $k=k+1$ .

Step 10. Verify the count ending condition  $k>N$ . If this condition is violated, return to step 3.

Step 11. End.

### **Control questions to the section**

1. For what pattern modeling the method of moments is intended?

2. For what pattern modeling the generalized J. Neumann exclusion method is intended?

3. What method should be used for multidimensional random variable modeling with a density function  $f(x) = 2x_1 e^{-\lambda x_2}$  ?

4. What method should be used for multidimensional random variable modeling with a density function  $f(x) = 2x_1 + x_2^2$ , where  $1 < x_1 < 5$ ,  $0 < x_2 < 3$ ?

## Chapter 6. Random processes modeling

The mathematical models of random processes in the general case are random functions of time  $\eta(t)$ . The main characteristics of the random function are the mathematical expectation  $M[\eta(t)] = m_x(t)$ , the dispersion  $D[\eta(t)]$ , which are non-random functions of time, and the correlation function  $R_x(t_i, t_j)$ , which is a non-random function of two variables  $t_i, t_j$ .

All these characteristics can be determined by processing the experimental data using methods of mathematical statistics.

### 6.1 A non-stationary random process modeling

For non-stationary random processes modeling academician V.S. Pugachev developed the canonical decomposition method [24].

Let take a random function  $\eta(t)$  with a correlation function  $R_x(t_i, t_j)$  and mathematical expectation  $m_x(t)$ , and a sequence of points  $t_1, t_2, \dots, t_n$  (not necessarily equally spaced from each other) on the time axis. It is required to obtain an implementation  $x(t)$  of the random process  $\eta(t)$ .

The principle of the canonical decomposition method is based on the following expression:

$$\eta(t) = m_x(t) + \sum_{i=1}^n v_i \varphi_i(t), \quad (6.1)$$

where  $v_i$  – centered uncorrelated random variables with a given distribution law,  $\varphi_i(t)$  – non-random functions  $t$  called coordinate functions.

The preparatory work for a random process modeling by the canonical decomposition method (6.1) comes to determining the coordinate functions and finding dispersions.

The canonical decomposition of the correlation function is written as:

$$R_x(t_i, t_j) = \sum_{k=1}^i \varphi_k(t_i) \cdot \varphi_k(t_j) \cdot D_k, \quad (6.2)$$

and the canonical decomposition of dispersion has the form:

$$D_x(t_i) = R_x(t_i, t_i) = \sum_{k=1}^i [\varphi_k(t_i)]^2 \cdot D_k. \quad (6.3)$$

Formulas for determining the coordinate functions  $\varphi_i(t)$  and dispersions  $D_i$  of random variables  $v_i$  follow from relations (6.2) and (6.3):

$$\varphi_i(t_j) = [R_x(t_i, t_j) - \sum_{k=1}^{i-1} D_k \varphi_k(t_i) \varphi_k(t_j)] / D_i. \quad (6.4)$$

It should be noted that at  $i > j$   $\varphi_i(t_j) = 0$ , and  $\varphi_i(t_i) = 1$ :

$$D_i = R_x(t_i, t_i) - \sum_{k=1}^{i-1} [\varphi_k(t_i)]^2 D_k. \quad (6.5)$$

And then, in order to calculate the implementation  $x(t)$  of the random process  $\eta(t)$ , we have the formula:

$$x(t_j) = m_x(t_j) + \sum_{i=1}^j y_i \cdot \varphi_i(t_j), j=1, n. \quad (6.6)$$

here  $y_i$  – implementations of random variables  $v_i$ . The distribution law of the random variable  $v_i$  can be arbitrary, in particular, uniform. The only exception is the Gaussian process, during the modeling of which the random variable  $v_i$  should have a normal distribution.

*Example 6.1.* Let it be necessary to simulate a normal non-stationary process defined by the mathematical expectation:

$$M_x(t) = a + bt$$

and correlation matrix:

$$R_x(t_i, t_j) = \begin{vmatrix} 0.35 & 0.26 & 0.20 \\ 0 & 0.30 & 0.24 \\ 0 & 0 & 0.26 \end{vmatrix}.$$

Using formulas (6.4) and (6.5), we calculate the coordinate functions and dispersions:

$$D_1 = R_x(t_1, t_1) = 0.35;$$

$$\varphi_1(t_1) = R_x(t_1, t_1) / D_1 = 0.35 / 0.35 = 1;$$

$$\varphi_1(t_2) = R_x(t_1, t_2) / D_1 = 0.26 / 0.35 = 0.743;$$

$$\varphi_1(t_3) = R_x(t_1, t_3) / D_1 = 0.2 / 0.35 = 0.571;$$

$$D_2 = R_x(t_2, t_2) - [\varphi_1(t_2)]^2 D_1 = 0.1068;$$

$$\varphi_2(t_1) = 0;$$

$$\varphi_2(t_2) = 1;$$

$$\varphi_2(t_3) = [R_x(t_2, t_3) - D_1 \cdot \varphi_1(t_2) \cdot \varphi_1(t_3)] / D_2 = 0.8568;$$

$$D_3 = R_x(t_3, t_3) - [\varphi_1(t_3)]^2 - [\varphi_2(t_3)]^2 * D_2 = 0.0675;$$

$$\varphi_3(t_1) = 0;$$

$$\varphi_3(t_2) = 0;$$

$$\varphi_3(t_3) = 1;$$

therefore, in order to calculate the implementations  $x(t_j)$  we have the formulas:

$$x(t_1) = a + bt_1 + y_1;$$

$$x(t_2) = a + bt_2 + 0.743y_1 + y_2;$$

$$x(t_3) = a + bt_3 + 0.571y_1 + 0.8568y_2 + y_3.$$

The algorithm of the canonical decomposition method consists of preliminary and main stages.

*Preliminary stage:*

Step 1. Calculation dispersion  $D_i$ ,  $i = \overline{1, n}$  and  $\varphi_i(t_j)$ ,  $i = \overline{1, n}$ ,  $j = \overline{1, n}$  by formulas (6.5) and (6.4).

*Main stage:*

Step 2. Obtain  $n$  implementations  $y_i$  of the random variable  $v$  with mathematical expectation  $m_y = 0$  and dispersion  $D_i$ .

Step 3. Calculate implementation  $x(t_j)$  by the formula (6.6).

Step 4. Accept  $j = j + 1$ .

Step 5. Verify the condition  $j > n$ , where  $n$  – a given number of points on the time axis. If this condition is violated, go to step 3.

Step 6. Display implementations  $x(t_j)$ .

## 6.2 Stationary random processes modeling

For stationary random processes modeling, it is necessary to specify the correlation function  $R(\tau)$ , mathematical expectation  $m$ , and dispersion  $\sigma^2$ . Then the formula for calculating the implementation of the stationary process at points  $t_1, \dots, t_n$  takes the form:

$$x(t_j) = m + \sum_{i=1}^n a_i y_{i+j-1}, \quad j = \overline{1, n}, \quad (6.7)$$

where  $y$  – implementations of uncorrelated random variables with mathematical expectation equal to zero and dispersion  $\sigma^2$ .

Coefficients  $a_i$  are calculated from relation:

$$R(t_k - t_1) = (a_1 a_k + a_2 a_{k+1} + \dots + a_{n-k+1} a_n) \sigma^2. \quad (6.8)$$

If the stationary random process is normal, then relations (6.7) and (6.8) take the following form:

$$x(t_j) = m + \sum_{i=1}^n a_i U_{i+j-1}, \quad j = \overline{1, n}, \quad (6.9)$$

$$R(t_k - t_1) = (a_1 a_k + a_2 a_{k+1} + \dots + a_{n-k+1} a_n), \quad (6.10)$$

where  $U$  – implementation of a normalized normally distributed random variable.

*Example 6.2.* It is required to obtain implementations  $x(t_j)$  of a normal stationary process with a correlation function of the following form:

$$R(\tau) = 0.2e^{-k\tau},$$

where,

$$\tau_l = t_{j+1} - t_j,$$

and with a given mathematical expectation  $m$  at points:

$$t_j = \{0, 1, 2, 3\}, \quad j = 1, 2, 3, 4.$$

*Solution.* Coefficients  $a_i$  are defined from the formula (6.10).

$$R(t_1 - t_1) = 0.2e^0 = a_1^2 + a_2^2 + a_3^2 + a_4^2;$$

$$R(t_2 - t_1) = 0.2e^{-2\tau_1} = a_1 a_2 + a_2 a_3 + a_3 a_4;$$

$$R(t_3 - t_1) = 0.2e^{-3\tau_1} = a_1 a_3 + a_2 a_4;$$

$$R(t_4 - t_1) = 0.2e^{-4\tau_1} = a_1 a_4.$$

Taking into account that at  $k > 24$   $0.2e^{-k\tau} \approx 0$ , we obtain a system of equations:

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = 0.2;$$

$$a_1 a_2 + a_2 a_3 + a_3 a_4 = 0.271;$$

$$a_1 a_3 + a_2 a_4 = 0;$$

$$a_1 a_4 = 0.$$

Let  $a_4 = 0$ ,  $a_1 \neq 0$ .

Then from the third equation we have:

$$a_1 a_3 = 0, \text{ i.e. } a_3 = 0.$$

Now the system of equations will take the form:

$$a_1^2 + a_2^2 = 0.2;$$

$$a_1 a_2 = 0.0271.$$

Solving it, we find:  $a_1 = 0.06; \quad a_2 = 0.44.$

Then we finally rewrite the formula (6.9) in the following form:

$$x(t_j) = m + 0.06U_j + 0.44U_{j+1}, \quad j = 1, 2, 3, 4.$$

The stationary process modeling algorithm consists of two stages.

*Preliminary stage:*

Step 1. Calculating the parameters  $a_i, i = 1, n$  from the relation (6.8).

*Main stage:*

Step 2. Obtain implementations  $y_1, y_2, \dots, y_{2n-1}$  of the random variable  $\nu$  with mathematical expectation equal to zero and dispersion  $\sigma^2$ .

Step 3. Accept  $j = 1$ .

Step 4. Calculate the implementation  $x(t_j)$  by the formula (6.7).

Step 5. Set  $j = j + 1$ .

Step 6. Verify the condition  $j > n$ . If this condition is violated, go to step 4.

Step 7. Display all implementations  $x(t_j)$ .

### 6.3 Markov processes modeling

Markov processes, as it is known [6], are called random processes, the prediction of the probabilistic characteristics of which in the future it is enough to know the state of the process at the moment.

Let consider a homogeneous Markov process (chain) with a countable number of states  $S_i, i = \overline{1, n}$  and discrete time.

For a complete definition of a homogeneous Markov chain, it is necessary to specify the initial probabilities  $P_i(0)$  and the transition probability matrix:

$$P = \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{pmatrix} \quad (6.11)$$

Here  $p_{ij} = P\{S_j/S_i\}$  – conditional probability of transition to state  $S_j$ , if at the previous step there was a state  $S_i$ .

A random process in a Markov chain is a random transition from one state  $S_i$  to another one  $S_j$  at fixed moments of time  $t_1, t_2, \dots$ , for example:

$$S_1 \rightarrow S_3 \rightarrow S_2 \rightarrow S_4 \rightarrow \dots \quad (6.12)$$

Therefore, the Markov process modeling consists in determining the chain (6.12) and is carried out according to the following principle [6]. Firstly, the initial state of the Markov chain is selected, which is given by the initial probabilities  $P_i(0)=p_{0i}, i = \overline{1, n}$ .

To do this, you can use the method of complete group of events modeling specified by the probability table:

$$\begin{pmatrix} S_1 & S_2 & \dots & S_n \\ P_{01} & P_{02} & \dots & P_{0n} \end{pmatrix} \quad (6.13)$$

and find the number of the initial state, for example,  $S_m$ . Similarly, the following state is found. However, here, elements of the  $m$ -th row of the matrix (6.11) are used as elements of the lower row of the table (6.13). As a result of repeated repetition of the indicated procedure, we obtain one of the possible implementations of the Markov process:

$$S_m \rightarrow S_k \rightarrow \dots \rightarrow S_l.$$

This principle can be used for more complex Markov processes modeling, for example, heterogeneous ones.

The algorithm for homogeneous Markov processes modeling with  $n$  discrete states consists of the following steps:

Step 1. Accept  $i=1$  and  $\kappa=0$ . here  $i$  – process implementation number;  $k$  – Markov chain state index.

Step 2. Set  $p_j=p_{kj}, j=\overline{1, n}$ .

Step 3. Obtain the implementation  $z$  of the basic random variable  $\xi$ .

Step 4. Accept  $k=1, R=p_k$ .

Step 5. Verify the condition  $z \leq R$ . If it is satisfied, go to step 7.

Step 6. Set  $\kappa=\kappa+1$  and  $R=R+p_k$ . Go to step 5.

Step 7. Set  $S_i=S_k, i=i+1$ .

Step 8. Verify the condition  $i > N$ , i.e. whether all implementations of the Markov process obtained. If not, return to step 2.

Step 9. Display the results obtained.

### Control questions to the section

1. For what pattern modeling the canonical decomposition method is intended?

2. Taxi at the end of the day may remain in good condition with a probability  $p=0.7$ , require minor repairs -  $p=0.2$ , major repairs -  $p=0.1$ . The probability of recovery after minor repairs is  $p=0.65$ , after major repairs is  $p=0.4$ . Then what form will the transition probability matrix have?

3. The communication channel has two states:  $S_0$  - free;  $S_1$  - busy. Find the Markov chain with the following data:

$$\begin{array}{c} S_0 \quad S_1 \\ S_0 \left| \begin{array}{cc} 0.2 & 0.8 \end{array} \right., S(t_0) = S_0, z_1=0.75, z_2=0.54, z_3=0.25. \\ S_1 \left| \begin{array}{cc} 0.3 & 0.7 \end{array} \right| \end{array}$$

## Chapter 7. Event flows modeling

### 7.1 Event flow properties

During the research of real systems using simulation modeling, often there is a need to generate random flows that imitate the patterns of a certain type of event: aircraft arriving at the airport, failures in the operation of a complex electronic device, calls at a telephone exchange, etc.

We give the following definition. A flow of events is a sequence of events that occur one after another at random moments of time.

The flow of events can be graphically represented as a sequence of moments  $t_1, t_2, \dots$  on the time axis  $0_t$  (Figure 7.1).

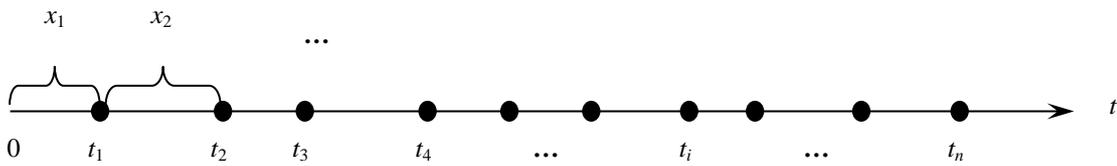


Figure 7.1 - A flow of events

Then the time intervals  $X_j$  between events in general case are implementations of the components of a vector random variable  $\eta = (\eta_1, \eta_2, \dots, \eta_n)$  with multidimensional distribution law  $F_n(x_1, x_2, \dots, x_n) = P\{\eta_1 < x_1, \eta_2 < x_2, \dots, \eta_n < x_n\}$

Event flows have various properties that allow to determine the types of flows [5].

7.1.1. First of all, flows can be homogeneous and heterogeneous. For example, the flow of messages received from journalists is heterogeneous, since messages marked by "lightning" are transmitted to the TV channel out of turn. However, in many cases, heterogeneous flows can be represented (Figure 7.2) as the superposition of several homogeneous flows.

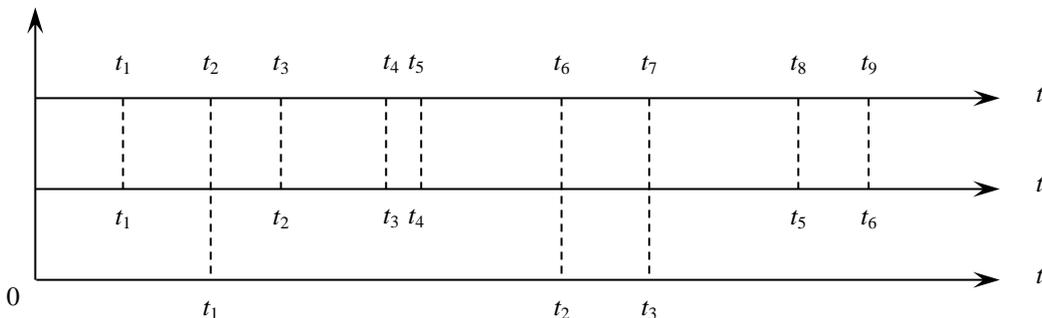


Figure 7.2 - Heterogeneous flow decomposition

7.1.2. The flow of events has the property of ordinarieness if the probability of the occurrence of two or more events during an elementary time interval  $\Delta t$  is

infinitely small compared to the probability of the occurrence of one event in this interval, i.e.:

$$P_1(t, t + \Delta t) \gg P_k(t, t + \Delta t), \quad k = 2, 3, \dots$$

Since for any interval  $\Delta t$ :

$$P_0(t, t + \Delta t) + P_1(t, t + \Delta t) + P_k(t, t + \Delta t) = 1, \quad k > 1,$$

then for an ordinary flow of events we have:

$$P_0(t, t + \Delta t) + P_1(t, t + \Delta t) \approx 1 \quad (7.1)$$

and

$$\lim_{\Delta t \rightarrow 0} \frac{P_k(t, t + \Delta t)}{\Delta t} = 0.$$

Examples of ordinary flows are a flow of aircraft arriving at the airport, a flow of shots at a target, etc. An example of an extraordinary flow of events can be the flow of passengers arriving at a given trolley bus stop.

7.1.3. The flow of events is called stationary if the probability of a certain number of events during a fixed-length time interval depends only on the duration of this interval and does not depend on its location on the time axis.

Let consider an ordinary flow of events on the time axis and find the average number of events occurring over the time interval  $\Delta t$ . In accordance with the expression (7.1), it will be:

$$0 \cdot P_0(t, t + \Delta t) + 1 \cdot P_1(t, t + \Delta t) = P_1(t, t + \Delta t).$$

Then the average number of events occurring per unit of time will be:

$$P_1(t, t + \Delta t) / \Delta t.$$

Let consider the limit of the last expression at  $\Delta t \rightarrow 0$ . If this limit exists, then it is called the intensity of an ordinary flow.

$$\lambda(t) = \lim_{\Delta t \rightarrow 0} \frac{P_1(t, t + \Delta t)}{\Delta t}.$$

The intensity  $\lambda(t)$  has a dimension opposite to time.

For a stationary flow, unlike a non-stationary flow, its intensity does not depend on time, i.e.

$$\lambda(t) = \lambda = \text{const}.$$

7.1.4. The flow of events has the property of no aftereffect if, for any two disjoint time intervals  $\tau_1$  and  $\tau_2$  (Figure 7.3) the number of events that fell on one of them does not depend on how many events fell on the other one. In essence, this means that the events that form the flow appear at certain moments of time independently of each other, each caused by its own causes. For example, the flow of people turning to the address bureau has practically no aftereffect, since there is no relationship between the causes that caused each of them to enter the bureau. But the flow of people who received the help already has a consequence, since each visitor needs an interval of time no less than the minimum time for writing the help.

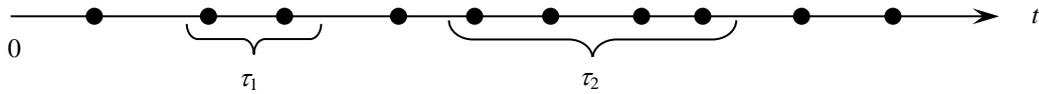


Figure 7.3 - A flow of events with no aftereffect property

7.1.5 A flow of events is called a flow with a limited aftereffect if the lengths of the intervals between its events  $\eta_1, \eta_2, \dots, \eta_n$  are mutually independent random variables and, therefore, their joint density function (under the assumption that  $\eta_j, j = \overline{1, n}$  - are continuous variables) can be represented as:

$$f_\eta(x_1, x_2, \dots, x_n) = f_1(x_1) * f_2(x_2) * \dots * f_n(x_n). \quad (7.2)$$

And for stationary flows with a limited aftereffect, the relation is also true:

$$f_1(x_1) \neq f_2(x_2) = f_3(x_3) = \dots = f_n(x_n) = f(x), \quad (7.3)$$

denoting the equality of the distributions of all intervals, except for the first one.

The meaning of the limited aftereffect is that the functions  $f_j(x_j)$  at  $j > 1$  are conditional density functions of the values  $\eta_j$  under the condition that an event occurs at the initial moment of the  $j$ -th interval ( $j > 1$ ). The only exception is the function  $f_1(x_1)$ , which is an unconditional density function of the first interval, since no assumptions are made regarding the appearance or non-appearance of the event at the initial moment of time.

## 7.2 Simplest flow modeling

The simplest flow is the Poisson flow of events, which, in addition to the properties of ordinariness and the absence of aftereffect, also has the property of

stationarity. The name "simplest" is due to the fact that the processes associated with these flows have the simplest mathematical description.

The simplest flow takes center stage among all the variety of flows, since at a sufficiently large number of independent flows (comparable in intensity) there is obtained a flow that is close to the simplest one.

As follows from the definition, the number of events of the simplest flow is a discrete random variable and obeys the Poisson distribution. Consequently, the probability of occurrence over a certain interval  $\tau$  of a certain number of events, for example  $k$ , is found by the formula:

$$P_k(\tau) = \frac{(\lambda\tau)^k}{k!} e^{-\lambda\tau} = \frac{a^k}{k!} e^{-a}, \quad (7.4)$$

where  $a = \lambda\tau$  – average number of events over time  $\tau$ .

Let find the distribution law of the interval  $\eta$  between any two neighboring events of the simplest flow:

$$F(x) = P\{\eta < x\}.$$

As follows from the figure 7.4, probabilities  $P\{\eta < x\}$  corresponds to the case when at least one simple flow event falls on the interval  $X$ , therefore:

$$F(x) = P\{\eta < x\} = 1 - P\{\eta \geq x\} = 1 - P_0(x) = 1 - e^{-\lambda x}. \quad (7.5)$$

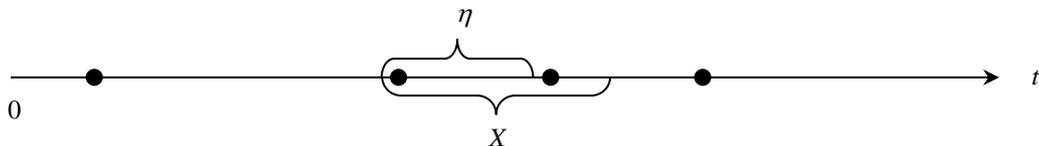


Figure 7.4 - Condition of the inequality  $\eta < x$

Differentiating (7.5), we obtain the distribution density of a random variable  $\eta$ :

$$f(x) = \frac{dF(x)}{dx} = \lambda e^{-\lambda x}, \quad (x > 0). \quad (7.6)$$

Thus, in the simplest flow, the interval between any two neighboring events is distributed according to the exponential distribution law with a parameter  $\lambda$ .

The result obtained is valuable in that it allows to use for simplest flow modeling, as well as any event flows, the apparatus previously studied for random variables modeling with a given distribution law. Indeed, from the fig. 7.1 it follows that the moments of random flow events occurrence can be found from the expression:

$$t_j = t_{j-1} + x_j,$$

where  $x_j$  – random variable implementation  $\eta_j$ , whose distribution law for the simplest flow has just been obtained by us.

The simplest flow modeling algorithm includes the following steps:

Step 1. Set  $j=1$ .

Step 2. Obtain the implementation  $z$  of the basic random variable  $\xi$ .

Step 3. Define the value of the interval between neighboring events of the simplest flow with intensity  $\lambda$ :

$$x = -\frac{1}{\lambda} \ln z .$$

Step 4. Calculate the moment of occurrence of the  $j$ -th event of the simplest flow  $t_j=t_{j-1}+x$ .

Step 5. Accept  $j=j+1$ .

Step 6. Verify the condition  $j>n$ . If this condition is violated, return to step 2.

Step 7. Display the result of modeling  $\{t_j\}$ .

### 7.3 Erlang flows modeling

Erlang flows owe their name to the Danish scientist who introduced them to study processes in telephone systems. Erlang flows have the properties of ordinarity, stationarity and limited aftereffect and are obtained by "sifting" a simple flow. If you save every second event in the simplest flow and discard the rest ones, you will get a second-order Erlang flow, and if every  $k$ -th event is saved, then a  $k$ -th Erlang flow is generated (Figure 7.5.)

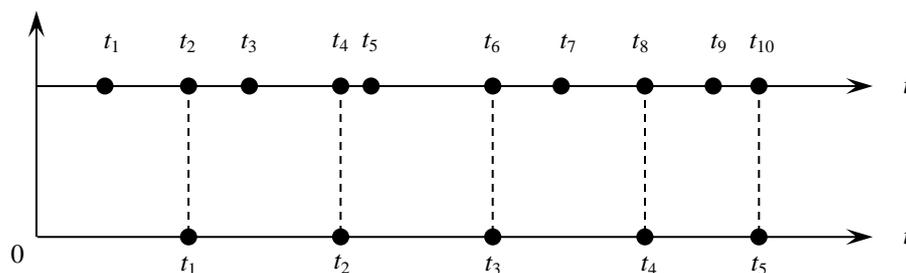


Figure 7.5 - A second-order Erlang flow formation

Obviously, the simplest flow is a special case of Erlang flow when  $k = 1$ .

The interval  $\eta$  between neighboring events of the Erlang flow of the  $k$ -th order is the sum of  $k$  independent random variables  $\eta_i$ , distributed according to the exponential law:

$$\eta = \sum_{i=1}^k \eta_i . \quad (7.8)$$

The density function of the random variable  $\eta$  is determined by the formula [19].

$$f_k(x) = \frac{\lambda^k}{(k-1)!} x^{k-1} e^{-\lambda x}, \quad x > 0, \quad (7.9)$$

where  $\lambda$  – simple flow intensity.

Mathematical expectation, dispersion, standard deviation of a random variable  $\eta$  respectively equal to:

$$\begin{aligned} M[\eta] &= M\left[\sum_{i=1}^k \eta_i\right] = \frac{k}{\lambda}; \\ D[\eta] &= D\left[\sum_{i=1}^k \eta_i\right] = \frac{k}{\lambda^2}; \\ \sigma_\eta &= \frac{\sqrt{k}}{\lambda}. \end{aligned} \quad (7.10)$$

Considering that the Erlang flow intensity of the  $k$ -th order is a value inverse to the mathematical expectation  $\Lambda_k = 1/M[\eta] = \lambda/k$ , we rewrite expressions (7.9) and (7.10) in the final form:

$$f_k(x) = \frac{(k\Lambda_k)^k}{(k-1)!} x^{k-1} e^{-k\Lambda_k x}, \quad x > 0 \quad (7.11)$$

$$M[\eta] = \frac{1}{\Lambda_k}, \quad D[\eta] = \frac{1}{k\Lambda_k^2}, \quad \sigma_\eta = \frac{1}{\Lambda_k \sqrt{k}}. \quad (7.12)$$

Let find out how the Erlang flow will behave if its order is unlimitedly increased ( $k \rightarrow \infty$ ), and the intensity is left unchanged ( $\Lambda_k = \text{const}$ ). From formulas (7.12) it can be seen that the mathematical expectation of the intervals between flow events remains constant, while the dispersion and standard deviation tends to zero, i.e. Erlang flow approaches the regular flow with constant intervals equal to  $1/\Lambda_k$ .

This property of Erlang flow has an important practical application. By various values of  $k$ , there can be obtained any degree of aftereffect: from the complete absence of aftereffect at  $k = 1$  to a rigid connection between the moments of events occurrence at  $k \rightarrow \infty$ . Therefore, many real flows of events can be approximated by the Erlang flow, choosing its order ( $k$ ) in such a way that the mathematical expectation and dispersion of both flows approximately coincide.

The Erlang flows modeling, which are quite satisfactory for practical purposes, is carried out according to the following algorithm:

Step 1. Set  $j=1$ .

Step 2. Set  $i=1, S=1$ .

Step 3. Obtain the implementation  $z$  of the basic random variable  $\xi$ .

- Step 4. Accept  $i=i+1$  and  $S=S \times z$ .
- Step 5. Verify the condition  $i>k$ . If it is violated, go to step 3.
- Step 6. Calculate the length of the interval between flow events and the moment the event occurrence  $x_j = -(1/\lambda) \ln(S)$ ,  $t_j = t_{j-1} + x_j$ .
- Step 7. Accept  $j=j+1$ .
- Step 8. Verify the condition  $j>n$ . If it is violated, go to step 2.
- Step 9. Display  $\{t_j\}$ .

#### 7.4 Palm flows modeling

The flow of ordinary events is called the Palm flow if the intervals between its consequent events are independent random variables  $\eta_j$ ,  $j=1, \dots, n$ , i.e. a limited aftereffect takes place. Therefore, for this flow, the following relation is true:

$$f(x_1, \dots, x_n) = f_1(x_1) \times f_2(x_2) \times \dots \times f_n(x_n),$$

which defines the general principle of Palm flow modeling. Using the corresponding density functions  $f_j(x_j)$ ,  $j=1, \dots, n$  it is necessary to find the implementations  $x_j$ ,  $j=1, \dots, n$  of all the intervals between the events of the flow and by the formula:

$$t_j = t_{j-1} + x_j$$

calculate desired moments  $\{t_j\}$ .

In practice, it is often necessary to deal with the stationary Palm flow, for which the relation is true:

$$f_1(x_1) = f_2(x_2) = f_3(x_3) = \dots = f_n(x_n) = f(x).$$

If the density function  $f(x)$ , which characterizes all the intervals, except the first one, is given, then the Palm formula is used to determine the unconditional density function  $f_1(x_1)$  [5]:

$$f_1(x_1) = \lambda \left(1 - \int_0^{x_1} f(x) dx\right), \quad (7.13)$$

where  $\lambda = 1/M[\eta]$ .

Therefore, in order to specify Palm flows, it is sufficient to know only the density function  $f(x)$ . As  $f(x)$ , there can be used functions that obey different distribution laws, including Erlang's law and exponential one. Consequently, Erlang flows, including the simplest flow, are special cases of the Palm flow.

The last remark may seem doubtful, since the simplest flow, obeying the absence of aftereffect property, has the same distribution for all intervals between its events, including the first one.

This contradiction is easily removed by the Palm formula (7.13), which turns out to be fair for a simplest flow with a density function:

$$f(x) = \lambda e^{-\lambda x}.$$

Indeed,

$$f_1(x_1) = \lambda(1 - \lambda \int_0^{x_1} e^{-\lambda x} dx) = \lambda e^{-\lambda x_1}.$$

The algorithm for stationary Palm flow modeling consists of two stages [19].

*Preliminary stage:*

Step 1. Using the given density function  $f(x)$ , calculate the mathematical expectation of the intervals  $\eta_j, j > 1$  and define Palm flow intensity  $\lambda = 1/M[\eta]$ .

Step 2. According to the Palm formula, find the distribution law of the first interval.

Step 3. Using the method of random variables conversion, for example, the inverse function method, obtain the dependencies:

$$x_1 = F_1^{-1}(z_1) \text{ and } x_j = F^{-1}(z_j), j > 1$$

to find implementations of the intervals between events of the Palm flow.

*Main stage:*

Step 1. Set  $j=1$ .

Step 2. Obtain the implementation  $z$  of the basic random variable  $\xi$ .

Step 3. Verify the condition  $j=1$ . If the condition is violated, go to step 5.

Step 4. Calculating  $x_1 = F_1^{-1}(z_1)$ . Go to step 6.

Step 5. Calculating  $x_j = F^{-1}(z_j), j > 1$ .

Step 6. Determining the moments of flow events occurrence  $t_j = t_{j-1} + x$ .

Step 7. Accept  $j=j+1$ .

Step 8. Verify the condition  $j > n$ . If the condition is violated, return to step 2.

Step 7. Display  $\{t_j\}$ .

In conclusion, we explain the clause made at describing the algorithm for Erlang flows modeling. Since the Erlang flow has the property of a limited aftereffect, the law of distribution of its first interval differs from the law of distribution of the remaining intervals and must be determined by the Palm formula. However, even for Erlang flows of the second order, this procedure, associated with the integration of the function (7.9), is very complex and requires approximate calculations. Therefore, in applied problems, the distinction between the distribution

laws of the first and other intervals between the Erlang flow events is usually neglected.

## 7.5 Modeling a non-ordinary flow of events

Let the flow of events be non-ordinary and the number of events occurring at time  $t_j$  be a discrete random variable with a given law of probability distribution:

$$v = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ p_1 & p_2 & p_3 & \dots & p_n \end{pmatrix}.$$

Then, in order to simulate the number of events that occurred at time  $t_j$ , there can be used the methods of discrete random variables modeling described earlier. And the moments  $t_j$  of the flow events occurrence are modeled in the same way as for the ordinary flows considered in this section.

Let present an algorithm for non-ordinary stationary Palma flow modeling.

*Preliminary stage:*

Step 1. For a given density function  $f(x)$  calculate the mathematical expectation of intervals  $\eta_j$  ( $j > 1$ ) and determine the intensity  $\lambda$  of the Palm flow.

Step 2. According to the Palm formula, define  $f_1(x_1)$ .

Step 3. Obtain dependences

$$x_1 = F_1^{-1}(z) \text{ and } x_j = F^{-1}(z_j), j > 1.$$

*Main stage:*

Step 4. Set  $j=1$ .

Step 5. Obtain the implementation  $z$  of the basic random variable  $\xi$ .

Step 6. Accept  $k=1$ .

Step 7. Verify the condition  $z \leq \sum_{i=1}^k p_i$ .

If this condition is met, go to step 9.

Step 8. Accept  $k=k+1$ . Return to step 4.

Step 9. For the next implementation of the random variable  $v$  we take the value  $k$ , i.e.  $v=k$ .

Step 10. Verify the condition  $j=1$ . If the condition is violated, go to step 12.

Step 11. Define the implementation of the first interval between flow events  $x_1 = F_1^{-1}(z)$ .

Step 12. Calculate  $x_j = F^{-1}(z_j)$ ,  $j \neq 1$ .

Step 13. Determining the moments of flow events occurrence  $t_j = t_{j-1} + x_j$ .

Step 14. Accept  $j=j+1$ .

Step 15. Verify the condition  $j > n$ . If the condition is violated, go to step 2.

Step 16. Display  $\{v, t_j\}$ .

### Control questions

1. What distribution characterizes the intervals between events of the simplest Poisson flow?

2. What distribution characterizes the intervals between Erlang flow events?

3. What distribution characterizes the intervals between events of the Palma flow?

4. Intervals between the moments of the appearance of hairdresser clients have a density function  $f(x) = \lambda e^{-\lambda x}$ ,  $x \geq 0$ . What kind of flow is it?

5. The intervals between the appearance of clients of the address bureau have a density function  $f(x) = \alpha^k [(k-1)!]^{-1} x^{(k-1)} e^{-\alpha x}$ . What kind of flow is it?

6. The intervals between the moments of the appearance of gas station customers have a density function  $f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{(x-m_x)^2}{2\sigma_x^2}}$ ,  $-\infty < x < \infty$ . What kind of flow is it?

7. The duration (min) [6 16] between the moments when clients arrive at the workshop has a uniform distribution. At  $t_1 = 8.20$  and  $z=0,20$  define the moment of arrival of the second client  $t_2$ .

8. The duration (min) between the moments when clients arrive at the hairdresser has a density function  $f(x) = \lambda(1 - (\lambda/2) \cdot x)$ ,  $x \in (0, 2/\lambda)$ . At  $\lambda = 0,1$ ,  $t_1 = 8.20$ ,  $z = 0,25$  define the moment of arrival of the second client  $t_2$ .

9. What is the name of the flow of events with the density function  $f(x) = \frac{1}{b-a}$ ?

10. What is the name of the flow of events with a density function of intervals between events  $f(x) = \frac{1}{b-a}$ ?

## **Part 2. ORGANIZATION AND APPLICATION OF COMPUTER MODELING**

### **Chapter 8. Computer modeling planning**

#### **8.1 Modeling stages**

Due to the variety of content of specific work on modeling systems, each computer modeling requires the sequential implementation of the following six stages:

- problem statement;
- creation of a mathematical model;
- compilation of a program for a computer;
- assessment of the adequacy of the model;
- experiment planning;
- interpretation of modeling results.

Let consider each stage separately.

##### **8.1.1 Problem statement**

Like any research, computer modeling should begin with the formulation of the modeling problem, i.e. with a clear statement of the objectives of the modeling and the limitations that must be considered at models creation. Objectives are usually formed in the form of either questions that need to be answered; or hypotheses that need to be verified; or impacts that need to be evaluated [19]. For example, computer modeling can be used to solve the following questions: how will the proposed algorithm for interrogating sensors affect the functioning of an automated technological process control system for a complex installation, or what is the impact of a specific procedure for operational planning of production? In addition, it is necessary not only to pose questions before modeling, but also to formulate criteria for evaluating possible answers to them.

The purpose of modeling can also be to test one or more hypotheses regarding the behavior of some complex system. How will the change in bus route affect the load on cars? Will the artificial migration of some animal species lead to disturbance of the ecological balance in the reserve? In each case, the tested hypothesis, as well as the criteria that decide to “accept” or “reject” it, must be formulated explicitly.

Finally, computer modeling can be undertaken to evaluate the impact of a control variable on input or dependent variables describing the behavior of the system. For example, it is necessary to evaluate the influence of the percentage of oxygen in the blast air on the  $SO_2$  content in the exhaust gases of metallurgical plants.

In order to determine the limitations of the modeling problem, it is necessary to identify the characteristics of the system under study. The first step along this

path is to establish boundary conditions, i.e. what is or is not a part of the system under study. Next, essential parameters and variable systems are determined that characterize the object of study and allow to establish the main limitations of the modeling problem.

### 8.1.2 Creation of a model

Under a mathematical model, we mean a set of relations that relate during the process time the characteristics of the process flowing in the system with its parameters, input signals, and initial conditions. The heterogeneous purpose of the elements of complex systems, the functioning under the influence of random factors lead to a variety of mathematical schemes used to describe complex systems and their elements. Among the mathematical schemes used in the study of complex systems, differential and difference equations, Markov processes, queuing systems, dynamical systems, aggregate systems, and probabilistic automata occupy a special place.

These schemes can be called typical mathematical schemes, since they are widely used in the study of complex systems. The presence of developed mathematical methods for studying these schemes significantly increases their value using elements of complex systems as models. However, there should be noticed the fact that any mathematical model is never adequate to the process under study, and reflects only its main features, taking into account the tasks facing the researcher. In this regard, the question arises about the complexity of the mathematical model. On the one hand, it can be argued that real systems are extremely complex, and therefore mathematical models that claim to describe the behavior of these systems should be quite complex if necessary. But this is true only to a certain extent, since creating models that are so complex that their implementation will require an exorbitantly much time does not make sense. It is necessary to create such mathematical models that would provide a fairly accurate description of the behavior of the system and would not require at the same time too much time for programming and calculations.

In computer modeling of complex systems, the mathematical model is converted into a modeling algorithm, with the help of which elementary phenomena that make up the process under study are imitated. At the same time, the logic structure, the sequence of flowing over time, the nature and composition of information about the states of the process are stored in the algorithm.

### 8.1.3 Compilation of a program for a computer

At this stage, before the developer of the model there is a problem of its description in a language acceptable for using a computer. A quick transition to computer modeling led to the development of a large number of specialized programming languages designed for this purpose. In practice, however, most of the proposed languages are oriented towards a certain mathematical scheme. For example, the GPSS language is well suited for the study of queuing problems, and the SIMULA language in the seventies of the last century was widely used to

simulate large economic systems, which are described by econometric models with a large number of equations.

The use of special languages instead of universal languages significantly saves programming time. Another advantage of special imitation languages is the presence in them of methods for detecting errors that significantly exceed the corresponding capabilities of universal languages.

It should be noted that universal languages are also widely used in simulation researches, since they have very necessary and useful properties for research. And if there is a problem of choosing a language, then the correctness of this choice, apparently, depends on the extent to which the researcher is fluent in modeling methods. In some cases, a simple language that is easier to understand and learn may be more valuable than any of the more "rich" languages that are harder to use due to its inherent features.

#### 8.1.4 Assessment of the adequacy of the model

In complex systems, which include objects of computer modeling, any model only partially reflects the real process. A model is considered good if, despite its incompleteness, it can accurately predict the effect of changes in the system on the overall efficiency of the system. Therefore, it is necessary to verify the degree of conformity (adequacy) of the model and the real process. There are three approaches for assessment of the adequacy. Applying the first of them, we must make sure that the model is true in the first approximation. For example, the question should be posed: will the model give absurd answers if its parameters take limit values.

The second approach to assessing the adequacy of the model is to verify the initial assumptions. For example, what parameters and model variables can be considered significant and whether all essential parameters of the object are covered by the model. The nature of the variable, i.e. whether it is significant or not, is usually determined by the influence of this variable on the criterion of system efficiency. In order to determine the degree of coverage in the model of all the essential parameters of an object, there are used modern methods of statistical analysis, such as analysis of the dispersion of efficiency criteria.

The third approach to assessing the adequacy of the model is to verify the transformation of information from input to output. This approach is based on the use of statistical samples to estimate average values and dispersion, dispersion analysis, regression analysis, factor analysis, spectral analysis, autocorrelation, tests using the agreement criteria, etc.

Finally, we should always remember about the consumer of information obtained by computer modeling. The development of a computer model cannot be justified if it does not benefit the consumer.

Taking all this into account, we formulate specific criteria that a good model should satisfy [19]. Such a model should be:

- simple and user friendly;
- convenient to manage;

- reliable in the sense of a guarantee against absurd decisions;
- complete in terms of the ability to solve the main problems;
- adaptive, making it easy to switch to other modifications or to update data.

#### 8.1.5 Experiment planning

After completing the stage of evaluating the suitability of the model, it is necessary to run (implement) the model in order to obtain the desired information. The modeling results obtained by reproducing the only implementation of the process described by the model due to the action of random factors cannot objectively characterize the process of model functioning. Therefore, the desired values in the study of processes by computer modeling are obtained as average values according to a large number of process implementations. An exception is the so-called ergodic processes, for which the desired values can be obtained by averaging over time the results of a single implementation of the process.

If the number of implementations  $N$  is large enough, then, by the law of large numbers, the obtained estimates become stable and can serve as approximate values of the desired random variables.

#### 8.1.6 Modeling results processing

After experimenting with the model, you need to process its results. For complex systems and a large number of implementations reproduced during modeling, the amount of information about the state of the system can be so significant that storing it in a computer's memory, processing and subsequent analysis is practically impossible, or, in any case, very time-consuming. Therefore, it is necessary in this way to organize the fixation and processing of the modeling results so that estimates of the desired values are formed gradually during the modeling, without special storage of all information about the system states.

If random factors are taken into account when modeling a certain system, then random values are also among the modeling results. In such a situation, average values, dispersions, and other probabilistic characteristics of the corresponding random variables obtained from the results of multiple modeling are used as estimates for the desired values.

The modeling stages discussed above are necessary for any other studies, for example, for system analysis and operations research. However, the implementation of individual stages in computer modeling has fundamental differences from other methods of systems research. Such stages include the stages of creating a mathematical model, planning an experiment, and interpreting modeling results. We will begin our acquaintance with these features of computer modeling with the stage of creating a mathematical model, in which such a feature is the need to develop modeling algorithms.

## 8.2 Principles of modeling algorithms creation

For computer modeling of a complex system, its mathematical model is transformed into a modeling algorithm, with the help of which there are imitated elementary phenomena that make up the process under study. Moreover, the algorithm retains the logical structure, the sequence of flowing in time, the nature and composition of information about the states of the process [5].

### 8.2.1 The $\Delta t$ principle

Let imagine the functioning of a complex system as a successive change of its state, described by the characteristics  $Z(t)=\{Z_1(t),Z_2(t),\dots,Z_n(t)\}$ . Obviously, the task of process modeling of system functioning is the creation of functions  $Z(t)$ , as well as the calculation of some values, which are indicators of the system functioning, by the values of this function. Let transform the relations of the mathematical model to such a way that it is convenient to calculate the values of  $Z(t+\Delta t)$  by the values of  $Z(\tau)$  taken at the moment of time  $\tau \leq t$ .

For deterministic processes, the approximate values of  $Z(t)$  at the moment of time  $t_0$ ,  $t_1 = t_0 + \Delta t_0$ ,  $t_2 = t_1 + \Delta t_1$ , etc. are obtained recursively, i.e. on  $Z(t_0)$  we find  $Z(t_1)$ , on  $Z(t_0)$  and  $Z(t_1)$ - $Z(t_2)$  and, in general case  $Z(t_1)=\varphi[Z(t_0),\dots,Z(t_{i-1})]$ . For random processes, the procedure differs only in that at each step the probability distribution for the possible states is calculated and the specific state is selected by lot. Thus, according to the probability distribution for the initial state, one of the possible values of  $Z(t_0)$  is chosen by lot, after which for the moment  $t_1=t_0+\Delta t_0$  there is constructed a conditional probability distribution of states under the condition  $Z(t_0)$ , and the state  $Z(t_1)$  is determined by lot, etc. As a result of this procedure, there is constructed one of the implementations of the multidimensional process of model functioning on the interval  $[t_0, T]$ .

The  $\Delta t$  principle is the most universal principle for modeling algorithms creation covering a very wide class of real complex systems. However, this principle is also the least economical in terms of computation time.

### 8.2.2 The principle of special states

During the study of many complex systems, two types of states can be distinguished: ordinary states in which the system is located almost all the time, and special states characteristic for the system at some isolated time intervals, coinciding with the moments when the input signals enter the system and the output of one of the characteristics  $Z_i(t)$  to the boundary of the existence area, etc. At these moments, the characteristics of the system  $Z_i(t)$  change abruptly, and the properties of the system are estimated from information on special states. For the described type of systems, modeling algorithms can be created according to the so-called principle of "special states". To apply this principle, the relations of the mathematical model are transformed to a form convenient for obtaining the next special state according to one or several previous special states. In this case, the

time moments of the special state are determined and the “idle” passes of the  $\Delta t$  principle are completely excluded.

### 8.2.3 The principle of sequential posting of applications

This principle is used for queuing systems modeling. Its essence is to trace the fate of individual applications in the order they are appeared in the system. Each application in the system goes through several stages of "processing", it can go to the queue or to a service device, wait for repair of a failed device, etc. The fate of the application is traced until a condition arises that impedes its further progress in stages: completion of service, receipt of a higher priority application in the system, etc.

This principle leads to very economical algorithms, but has a rather complicated logical structure.

It should be noted that in practice often modeling algorithms are based on several principles simultaneously. For example, the general structure of the modeling algorithm is based on the principle of special states, and the principle of sequential posting of applications is used between special states.

## 8.3 General structure of the modeling algorithm

The general structure of the algorithm should ensure the achievement of the main modeling task - finding the optimal variant or the optimal values of the system parameters. To achieve this goal, it is necessary, firstly, to be able to simulate the process of system functioning at any given time interval, secondly, to ensure the required accuracy and reliability of the obtained values of the characteristics and indicators of the system, taking into account various random factors and, thirdly, to be able to search for optimal system variants [22].

These requirements predetermine the general structure of the modeling algorithm, including three levels or three modeling cycles (Figure 8.1). The inner cycle (blocks 5-8) allows to simulate the behavior of the system according to a given model on the interval  $[0, T]$ . The work of the model in this interval will be called a run.

In the next cycle, which includes blocks 3-10, there is organized an  $N$ -fold repetition of the run, which allows, after appropriate statistical processing of the results (block 11), to judge the averaged characteristics of the modeled version of the system. The end of a variant can be determined not only by a given number of runs, as shown on the figure 9.1 (block 10), but also by a given accuracy of the results.

The external cycle covers both previous cycles and additionally includes blocks 1, 2, 11, 12, which control the sequence of modeling system variants. Here, in particular, is organized the search for optimal system parameters: block 11 checks whether the system performance is satisfactory, and block 1 makes changes to the parameters in order to improve these indicators.

It should be noted that in many cases, when there is no need to consider various variants of the system, we can use the two-level structure of modeling algorithms.

## 8.4 Experiment planning

The main feature of the experimentation stage in computer modeling is that, due to the action of various random factors, the modeling results are also random in nature. Therefore, to ensure the given accuracy of the obtained estimates, it is necessary to average the results of a large number of implementations. To determine the number of implementations, it is necessary to establish a relationship between the accuracy of the estimate, the number of implementations, and confidence probability. Let find a quantitative relationship between the indicated values for various estimates of random variables and processes.

8.4.1 Determination of the number of implementations for probability estimation

Probability  $\bar{p}$  is estimates by the relation  $p$

$$\bar{p} = \frac{m}{N},$$

where  $m$  - the number of cases of the  $A$  event occurrence at  $N$  implementations. This relation can be represented in a different form [34]

$$\bar{p} = (1/N) \sum_{i=1}^N \xi_i, \quad (8.1)$$

where  $\xi_i$  – discrete random variable with distribution law

$$\xi_i = \begin{vmatrix} 1 & 0 \\ p & p-1 \end{vmatrix}.$$

Define the mathematical expectation and dispersion of the estimate  $p$

$$M[(1/N) \sum_{i=1}^N \xi_i] = (1/N) \sum_{i=1}^N M[\xi_i] = N \cdot p / N = p, \quad (8.2)$$

$$D\{(1/N) \sum_{i=1}^N \xi_i\} = (1/N^2) \sum_{i=1}^N D[\xi_i] = (1/N^2) N \cdot p(1-p) = p(1-p)/N.$$

Next, we establish the relationship between the accuracy of the estimate  $\varepsilon$ , the number of implementations  $N$ , and the confidence probability  $\alpha$ :

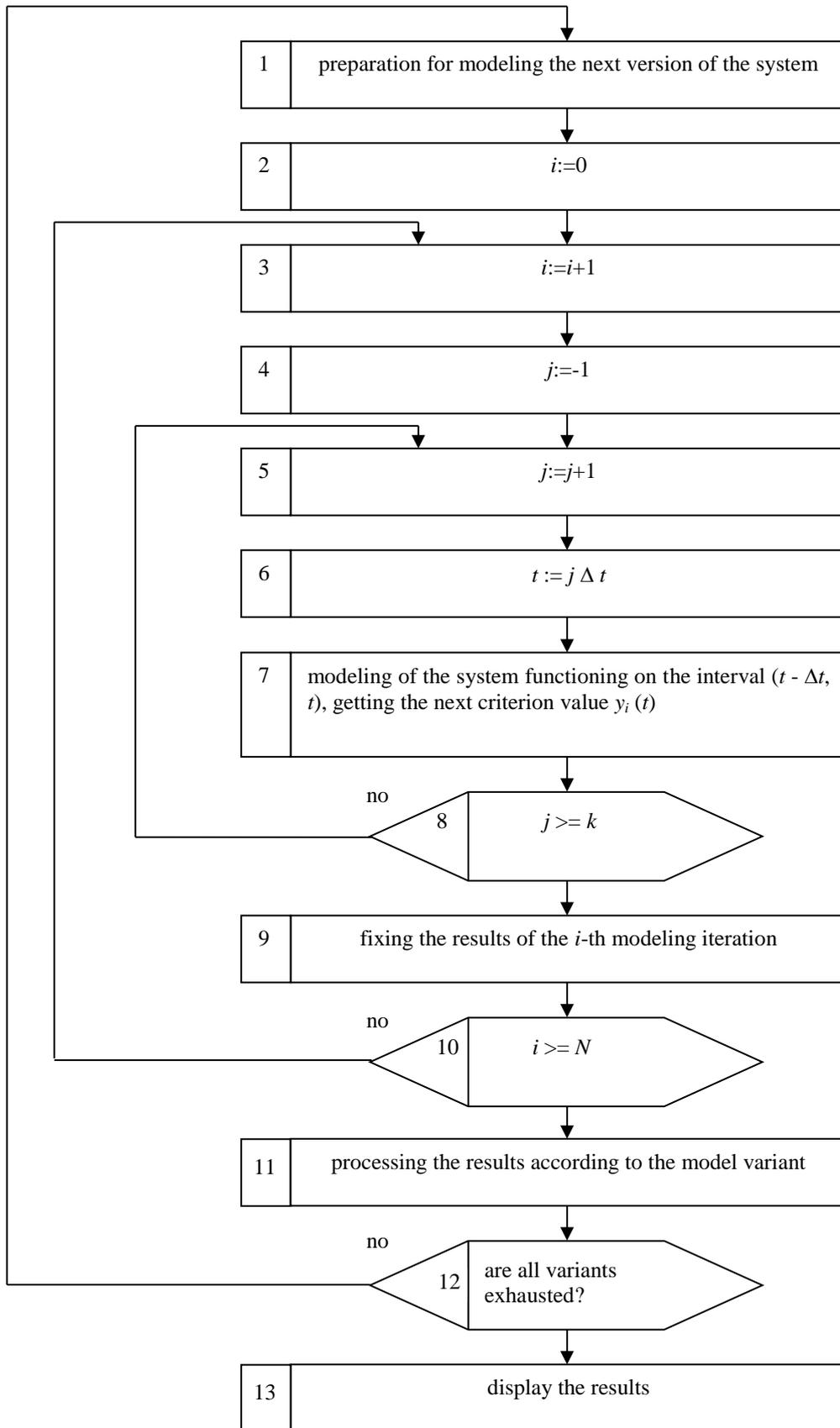


Figure 8.1 - General structure of the modeling algorithm

$$P\{[m/N - p] < \varepsilon\} = \alpha.$$

Due to the central limit theorem of probability theory, estimate (8.1) at sufficiently large  $N$  has a distribution close to normal. Therefore, from the normal distribution table [6] for each value of the confidence probability  $\alpha$ , there can be chosen a value  $t_\alpha$  that the accuracy of the estimate is equal to

$$\varepsilon = [m/N - p] = t_\alpha \sqrt{D}. \quad (8.3)$$

Substituting (8.2) into (8.3), we obtain the desired expression for the amount of implementations in the probability estimate

$$N = t_\alpha^2 p(1-p) / \varepsilon^2. \quad (8.4)$$

It follows from (8.4) that, in order to determine  $N$ , it is necessary to know the values of the estimated probability  $p$ , which is usually unknown. Therefore, they almost always make a preliminary “shooting”, assigning any number of implementations  $N_0$  and determining  $p_0 \approx k/N_0$ . Then, the found  $p$  is substituted into (8.4).

8.4.2 The selection of  $N$  at evaluating the mathematical expectation and dispersion.

To estimate the mathematical expectation of a random variable, there is used the arithmetic mean

$$\bar{X} = (1/N) \sum_{i=1}^N X_i.$$

Let it be necessary to create an estimate  $\bar{X}$  of the true mathematical expectation  $m$  such that:

$$P\{m - \varepsilon \leq \bar{X} \leq m + \varepsilon\} = 1 - \alpha.$$

According to the central limit theorem at large  $N$ , the arithmetic mean  $\bar{X}$  has a distribution close to normal with a mathematical expectation  $m$  and dispersion of  $\sigma^2/N$ , where  $\sigma^2$  is the dispersion of the estimated random variable. Therefore, substituting the dispersion  $\sigma^2$  in a formula similar to (9.3), we have

$$\varepsilon = [X - m] = t_\alpha \cdot \sigma \sqrt{N},$$

consequently,

$$N = t_\alpha^2 \cdot \sigma^2 / \varepsilon^2. \quad (8.5)$$

Since  $\sigma^2$  is unknown in advance, then instead of it we use the estimate  $S^2$  obtained by the formula:

$$S^2 = (1/N_0 - 1) \sum_{i=1}^N (X_i - \bar{X}),$$

as a result of the preliminary “shooting” with the number of implementations  $N_0$ .

We determine the number of implementations  $N$  required to obtain the dispersion estimate. It is assumed that:

$$P\{(1-\varepsilon) \cdot \sigma^2 \leq S^2 \leq (1+\varepsilon) \cdot \sigma^2\} = 1-\alpha,$$

where  $0 \leq \varepsilon \leq 1$  – a number characterizing the degree of proximity of the estimate  $S^2$  to true dispersion  $\sigma^2$ .

Since  $S^2$  is distributed asymptotically normally with parameters [6]:

$$m[S^2] = (N - \frac{1}{N})S^2, \quad D[S^2] = \frac{(m_4 - \sigma^4)}{N} + 0(\frac{1}{N^2}),$$

then, neglecting  $0(1/N^2)$  and taking into account that in the case of a normal distribution of the estimated random variable  $m_4 = 3\sigma^4$ , we obtain:

$$\varepsilon = [S^2 - \sigma^2] = t_\alpha \sqrt{\frac{2\sigma^4}{N}}.$$

Consequently,  $N = 2t_\alpha^2 \sigma^4 / \varepsilon^2$ .

In order to determine  $\sigma$ , there is used a “shooting”.

#### 8.4.3 Application of Chebyshev inequality for the selection of $N$

Until now, during the determination of the number of implementations, we proceeded from the assumption that the estimated parameter belongs to the class of random variables with a normal distribution law. If this assumption is rejected in order to determine  $N$  we can use the Chebyshev inequality, which has the form

$$P\{[\bar{X} - m] > k\sigma\} \leq 1/k^2. \quad (8.6)$$

This inequality shows that for a given number  $k$  and an arbitrary sample of  $X_1, X_2, \dots$  of size  $N$ , the probability of deviations of the measurements from the average value by a distance of more than  $k\sigma$  is less than or equal to  $1/k$ .

Example [34]. Let suppose that it is necessary that the estimate of the average value of a random variable falls into the interval  $m \pm \sigma/4$  with a probability of 0.95, i.e.

$$P\{[\bar{X} - m] > \sigma/4\} \leq 0.05.$$

Using equation (9.6) and taking into account that the standard deviation of the mathematical expectation estimate is  $\sigma/\sqrt{N}$ , we obtain:

$$P\{[\bar{X} - m] > N/4 \cdot \sigma/\sqrt{N}\} \leq 0.05 = 16/N, \quad K = \sqrt{N/4}, \quad 1/K^2 = 16/N,$$

Consequently,  $N=320$ .

The resulting volume of implementation is significantly larger than that which is sufficient in case of a normal distribution of the relation. However, it allows to obtain guaranteed accuracy with deviations of the random variable distribution from normal one.

## 8.5 Regenerative method for the analysis of modeling results

As noted earlier, the stage of processing the results of computer modeling also has its own peculiarity. It is connected with the fact that many modeling objects have the property of being regenerated from time to time, i.e. stochastic processes taking place in such systems constantly return to a certain point (regeneration point), starting from which the further development of the process does not depend on its behavior in the past and is determined by the same probabilistic law. If modeling results of such systems are grouped in accordance with successive returns to the regeneration point, then these groups are statistically independent and equally distributed, that greatly facilitates their statistical analysis [14].

In this section we give a brief summary of the regenerative method and show its main advantages in an illustrative example.

### 8.5.1 Regenerative method

A sequence  $\{X_n\}$  of random vectors of dimension  $k$  is a regenerative process, if there is an increasing sequence:

$$1 \leq \theta_1 < \theta_2 < \dots$$

of random discrete moments of time, called regeneration moments, is such that the process, starting from each of these moments, is determined by the same probabilistic laws as at time  $\theta_l$ . The part of the process  $\{X_n\}$  in the interval  $\theta_j \leq n \leq \theta_{j+1}$  will be called the  $j$ -th cycle. We also introduce the concept of the regeneration period  $\alpha_j$

$$\alpha_j = \theta_{j+1} - \theta_j.$$

These periods are independent and equally distributed quantities.

Let further,

$$Y_j = \sum_{i=\theta_j}^{\theta_{j+1}-1} f(X_i),$$

where  $f(X_i)$  – some function of  $k$  arguments taking real values. By an appropriate selection of the function  $f$ , a fairly wide class of stationary characteristics of practical interest can be estimated.

Further, we assume that the purpose of modeling is to estimate the value of  $M\{f(X)\}$ .

Since  $Y_j$  is the sum of the values of  $f(X_i)$  on the  $j$ -th cycle ( $j$ -th group), then the sequence  $\{Y_j, \alpha_j\}$  consists of independent and identically distributed random vectors. Under the rather weak assumption that:

$$M\{(f(X))\} < \infty,$$

we can obtain an expression:

$$M\{f(X)\} < \frac{M\{Y_j\}}{M\{\alpha_j\}}. \quad (8.7)$$

Indeed,

$$\frac{(f(X_1) + \dots + f(X_n))}{N} \rightarrow M\{f(X)\} \quad (8.8)$$

at  $N \rightarrow \infty$ .

And if the  $n$ -th cycle ends at a moment of time  $N$ , then the relation in (8.8) can be written differently:

$$((Y_1 + \dots + Y_n)/n)/(\alpha_1 + \dots + \alpha_n)/n. \quad (8.9)$$

But relation (8.9) with probability 1 converges to  $M\{Y_1\}/M\{\alpha_1\}$  at  $n \rightarrow \infty$ . And, therefore, we obtain the relation (8.7). Thus, we can state that the estimation problem of  $M\{f(X)\}$  is the same as the estimate of the relation  $M\{Y_1\}/M\{\alpha_1\}$ , which, based on classical statistics, can be estimated by independent and equally distributed pairs:

$$\{Y_1, \alpha_1\}, \dots, \{Y_n, \alpha_n\}.$$

### 8.5.2 Regenerative method application

Let illustrate the basic idea and advantages of the regenerative approach on the example of a shoe shiner (Section B.3). For the quality criterion of the system we take the average waiting time for the client for the service  $M\{\tau^{occ}\}$  in the steady state. Thus, the purpose of system modeling is to obtain an estimate of  $M\{\tau^{occ}\}$  with

a given "reliability", for example, with a 90% confidence interval of the true value of  $M\{\tau^{o,nc}\}$ .

As it is known [36], the traditional way to determine the estimate of  $M\{\tau^{o,nc}\}$  is to find a sample mean of  $m$ :

$$\bar{m} = \frac{1}{N} \sum_{k=1}^N \tau_k . \quad (8.10)$$

However, this method has at least two significant disadvantages. Firstly, due to the initial conditions an expression (8.10) gives a biased estimate for the sample mean. Indeed, if  $M\{\tau_i^{o,nc}\}=0$ , then the first few values of the waiting time tend to be small (Table B.1). An obvious, at first glance, way of eliminating the influence of the initial bias is to discard the waiting time of the first  $N_0$  requests in formula (8.10) in order to achieve a stationary mode of the system, i.e. using the formula:

$$M\{\tau^{o,nc}\} = \frac{1}{N} \sum_{k=N_0+1}^{N_0+N} \tau_k^{o,nc}$$

as an estimate of  $M\{\tau^{o,nc}\}$  [14]. But in this case, machine time is unproductive, since the value of  $N_0$  is not known in advance and it is taken quite large. Secondly, the sample values of the waiting times  $\tau_i^{o,nc}$  and  $\tau_j^{o,nc}$  in practice are strongly correlated, which casts doubt on the validity of the estimate  $M\{\tau^{o,nc}\}$ , obtained from (8.10) under the assumption of the statistical independence of identically distributed sample values. Indeed, if  $\tau_i^{o,nc}$  is large, then the next  $(i+1)$ -th request will, as a rule, also has a long waiting time, and vice versa (Table B.1).

Consequently, the bias caused by the initial conditions and the strongly correlated input data pose serious obstacles to using the estimation procedure based on the sample mean.

Let show how with the help of the regenerative approach these obstacles can be overcome in relation to this example. For this information the tab. B.1 will be presented in the form of the following graph (Figure 8.2).

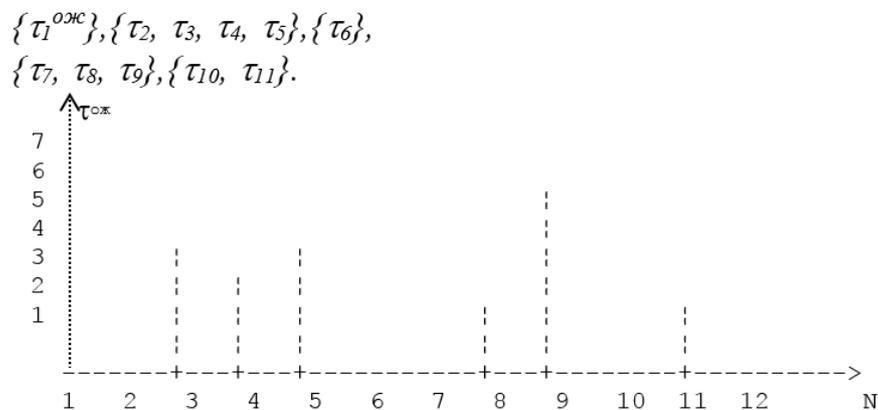


Figure 8.2 - Full cycle graph

From this figure it can easily be established that the period of time, consisting of the period of busyness and the subsequent period of downtime, is a cycle of the regenerative process. And the moments of arrival of customers 1, 2, 6, 7, 10, 12, catching the cleaner free, are the points of regeneration. Therefore, the graph of this example has five complete cycles:

The sixth cycle begins with the arrival of the 12<sup>th</sup> customer. Since each cycle begins with the restoration (regeneration) of the system to the state that was when the first client arrived, the data groups of successive cycles are statistically independent and have the same distribution.

In this example, we assume that  $Y_j$  is the sum of the waiting times values on the  $j$ -th cycle, and  $\alpha_j$  is the number of clients served on the  $j$ -th cycle.

Therefore, the strongly correlated data  $\{\tau_1, \tau_2, \dots, \tau_{11}\}$  were divided into statistically independent and identical groups, each of which is characterized by a pair  $\{Y_j, \alpha_j\}$ .

In order to estimate the average waiting time for a client, we use a formula similar to (8.7),

$$m = M\{\tau^{o.c.}\} = M\{Y_1\} / M\{\alpha\}.$$

For a better understanding of the above mentioned, we substitute specific numerical data. Suppose that it is necessary to obtain a 90% confidence interval for the average customer waiting time  $M\{\tau^{o.c.}\}$  in the stationary mode of the shoe shiner. The specific values of  $\tau^{o.c.}$  for each client are given in table. B1. In addition, from the fig. 9.2 it follows:

$$\begin{aligned} \{Y_1, \alpha_1\} &= \{0, 1\}, & \{Y_4, \alpha_4\} &= \{6, 3\}, \\ \{Y_2, \alpha_2\} &= \{8, 4\}, & \{Y_5, \alpha_5\} &= \{1, 2\}, \\ \{Y_3, \alpha_3\} &= \{0, 1\}. \end{aligned}$$

Find the values of sample means of  $Y_{cp}$  and  $\alpha_{cp}$ :

$$\bar{Y}_{cp} = (1/n) \sum_{j=1}^n Y_j = 1/5(0+8+0+6+1) = 3;$$

$$\bar{\alpha}_{cp} = (1/n) \sum_{j=1}^n L_j = 1/5(1+4+1+3+2) = 11/5.$$

The sample dispersions from the values  $Y_j, \alpha_j$  and the second sample mixed moment from the values of  $(Y_j, \alpha_j)$  are respectively equal to:

$$S_{11} = (1/n-1) \sum Y_j^2 - (1/n(n-1)) (\sum_{j=1}^n Y_j^2) = 14;$$

$$S_{22} = (1/n - 1) \sum_{j=1}^n (\alpha_j^2 - (1/n(n-1)) (\sum_{j=1}^n \alpha_j^2)) = 1.7;$$

$$S_{12} = (1/n - 1) \sum_{j=1}^n Y_j \alpha_j - (1/n(n-1)) (\sum_{j=1}^n Y_j) (\sum_{j=1}^n \alpha[j]) = 19/4.$$

Then  $S^2 = S_{11} - 2mS_{12} + S_{22} = 4.2$ .

Confidence interval

$$I = m \pm t_{0,1} \cdot S / (\alpha \cdot \sqrt{n}) = 15/11 \pm 0,6858,$$

the length of  $I$  is equal to 1,3716.

The last relation shows that the reduction of it at a given level of confidence would require additionally about 20 cycles.

## Chapter 9. DISTRIBUTION LAWS IDENTIFICATION

All previous sections of this paper have been devoted to methods for modeling various random patterns in the assumption that they are known. However, in solving practical problems, it is necessary to establish (identify) the patterns attributable to various parameters or variables having a statistical nature and significantly affecting the functioning of the objects under study.

Taking into account the focus of this work on disclosing the problem of random patterns modeling, we will cover the issues of identifying distribution laws quite briefly and without rigorous mathematical justification. The emphasis here will be placed on the consideration of the applied aspects of this issue.

The initial material for identifying the distribution laws is the experimental data obtained as a result of a large number of observations and forming a sample of a given volume.

### 9.1 Identification of the numerical characteristics of the sample

The arithmetic mean, or simply mean, is one of the main characteristics of the sample, with the help of which mathematical expectation is described (or identified). Mean, like other numerical characteristics of the sample, can be calculated both on unprocessed primary data and on the results of grouping these data [34]. The calculation accuracy on unprocessed data is always higher, but the calculation process is laborious at a large sample size.

For ungrouped data, the arithmetic mean is determined by the following formula:

$$\bar{x} = (1/n) \sum_{j=1}^n x_j, \quad (9.1)$$

where  $n$  – sample size,  $x_j$  – sample elements.

If data is grouped, then:

$$\bar{x} = (1/n) \sum_{i=1}^k n_i x_i, \quad (9.2)$$

where  $n_i$  – interval frequencies,  $k$  – number of grouping intervals,  $x_i$  – average values of intervals.

The arithmetic mean calculated by the formula (9.2) is also called the weighted average, since  $x_i$  is summed with weights equal to the frequencies of falling into the grouping intervals.

In order to identify the dispersion, there is used a sample dispersion  $S^2$ , which is calculated by the following formulas:

$$S^2 = \frac{1}{n-1} \sum_{j=1}^n (x_j - \bar{x})^2 \quad (9.3)$$

for ungrouped data;

$$S^2 = \frac{1}{n-1} \sum_{j=1}^n n_j (x_j - \bar{x})^2 \quad (9.4)$$

for grouped data.

Formulas (9.3) and (9.4) are inconvenient from a computational point of view, therefore, in practice, other calculation formulas are used, more convenient for both manual calculations and computer calculations.

For ungrouped data:

$$S^2 = \frac{1}{n-1} \left( \sum_{j=1}^n x_j^2 - n\bar{x}^2 \right) \quad (9.5)$$

or

$$S^2 = \frac{1}{n-1} \left[ \sum_{j=1}^n x_j^2 - \left( \sum_{i=1}^n x_i \right)^2 / n \right]. \quad (9.6)$$

If data is grouped, then:

$$S^2 = \frac{1}{n-1} \left( \sum_{j=1}^n n_j x_j^2 - n\bar{x}^2 \right) \quad (9.7)$$

or

$$S^2 = \frac{1}{n-1} \left[ \sum_{j=1}^n n_j x_j^2 - \left( \sum_{i=1}^n n_i x_i \right)^2 / n \right]. \quad (8.8)$$

Formulas (9.5) and (9.7) are used to determine the dispersion, if  $\bar{x}$  is already calculated. Formulas (9.6) and (9.8) are used in cases when  $\bar{x}$  and  $S^2$  is calculated simultaneously.

In order to identify the correlation coefficients between a pair of random variables, there is used a sample Bravais-Pearson correlation coefficient [21]:

$$r = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\left[ \sum_{i=1}^n (x_i - \bar{x})^2 \right] \cdot \left[ \sum_{i=1}^n (y_i - \bar{y})^2 \right]}}$$

For practical calculations, more convenient is the following formula:

$$r = \frac{n \sum_{i=1}^n x_i y_i - \left( \sum_{i=1}^n x_i \right) \cdot \left( \sum_{i=1}^n y_i \right)}{\sqrt{\left[ n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2 \right] \cdot \left[ n \sum_{i=1}^n y_i^2 - \left( \sum_{i=1}^n y_i \right)^2 \right]}}$$

The values of the correlation coefficient obtained from this formula are within the limits :

$$-1 \leq r \leq +1.$$

## 9.2 Identification of density functions of continuous random variables

The process of density functions identification can be conditionally divided into six stages.

The first of them is the grouping of elements of a given sample of the size  $n$ , i.e. the entire range of sample element values is subdivided into a certain number of disjoint intervals. The number of grouping intervals  $k$  depends on the sample size  $n$ .

Let give some recommendations for selecting the number  $k$  [21].

Selecting  $k$ , it should be taken into account that if its value is too large, the distribution pattern is distorted by random zigzags of frequencies that are too small at narrow intervals. If the number of intervals is too small, the characteristic features of the distribution are smoothed out and obscured. Therefore, multivariate calculations with a different number of intervals are desirable.

In order to determine the  $k$  number, you can use, for example, the following expressions:

- 1)  $k = 1 + 3.32 \lg(n)$  (Sturges' formula);
- 2)  $k = 5 \lg(n)$  and  $67 \leq k \leq 20$ ;
- 3)  $n/k = 50/8, 100/10, 500/13, 1000/15, 10000/20$ ;
- 4)  $k = \min(\sqrt{n}, 30)$ .

The next step is to determine the length and boundaries of the grouping intervals. If the grouping intervals are the same, then their length is calculated according to the formula [21]:

$$d = 1.02(X_{\max} - X_{\min})/k,$$

where  $X_{\max}, X_{\min}$  – maximum and minimum sample values.

The boundaries of the individual intervals are defined as follows:

$$[X_{\min} + (j-1)\alpha - \Delta, X_{\min} + j\alpha - \Delta], \quad j = \overline{1, k},$$

where  $j$  – interval number, and  $\Delta = 0.01\alpha$ .

The lengths of the grouping intervals can also be selected, according to another principle from the condition of the probability that the implementations of a random value fall into all intervals. During the creation of a partitioning scheme based on some expected distribution law, it is necessary to consider not the range of values specified by the sample set, but the theoretical range of possible values of a random variable with the expected distribution law.

At the third stage of distribution density identification, the relative frequencies  $v_j = n_j/n$ ,  $j = \overline{1, k}$  of the observations that fall into all intervals are calculated, and a histogram of the data is created (distribution of relative frequencies).

The fourth stage is the most responsible. Here, by the form of the obtained histogram, theoretical distributions are selected that are suitable for this case, which there may be several.

At the fifth stage, by comparing selective numerical characteristics (arithmetic mean, selective dispersion) with their theoretical values, the most suitable theoretical distribution is selected.

However, the final result of the distribution law identification can be obtained only at the sixth stage based on the results of the verification of the selected distribution using one of the consent criteria.

### **9.3 Identification of the distribution laws of discrete random variables**

The identification scheme for the distribution laws of discrete random variables does not have fundamental differences from the scheme for continuous quantities. The unprincipled differences of these schemes include the following ones:

1. The number of  $k$  groups here coincides with the number of all possible values of a discrete random variable. Therefore, the identification scheme begins with the third stage.

2. If it is impossible to select a suitable theoretical distribution law, we can use only the tabular form of setting the distribution law of a discrete random variable. As estimates of the probabilities, the relative frequencies of occurrence of each of the possible values of a discrete random variable are used.

### **9.4 Identification Results Assessment**

For statistical evaluation of the hypothesis that the set of empirical data differs insignificantly from that which can be expected with the selected theoretical distribution law, the most commonly used criteria are the consent criteria of Pearson, Kolmogorov-Smirnov, Mises. Each of these criteria has its own strengths and weaknesses and only the most general recommendations can be made regarding the choice between them.

Pearson criterion is effective for large sample sizes ( $n > 100$ ). The Kolmogorov-Smirnov criterion gives good results at  $10 \leq n \leq 100$ . With a sample size

of less than 10, satisfactory results can be obtained, perhaps, only with the help of the Mises criterion. Using the Pearson and Kolmogorov-Smirnov criteria, it is necessary to specify the number of grouping intervals. In case of the Pearson criterion, this number is determined from the condition that at least 5 experimental points fall in each interval. At the same time, in case of using the Kolmogorov-Smirnov criterion, the data can be both grouped and each observation can be assigned to a separate group. This condition opens up the possibility of efficient analysis for small samples.

Considering that the greatest distribution among consent criteria was given to Pearson criterion (criterion  $\chi^2$ ), we will consider it in more detail.

Let there be a sample of  $n$  implementations  $x_i, i = \overline{1, n}$  of a random variable  $\eta$  with the desired distribution law. Based on the results of the grouping of elements of a given sample, all relative frequencies  $\nu_j, j = \overline{1, k}$  and the corresponding probabilities  $p_j, j = \overline{1, k}$  of a selected theoretical (hypothetical) distribution have already been calculated.

The hypothesis is put forward that the random variable  $\eta$  really has a distribution law selected as a result of identification. In order to verify this hypothesis, we must choose a measure of the discrepancy between the statistical distribution and the hypothetical one.

As such a measure  $R$  using Pearson's criterion, we take the sum of squared deviations  $(\nu_j - p_j)$  of statistical frequencies from hypothetical probabilities  $p_j$  taken with some weights  $c_j$  [34]:

$$R = \sum_{j=1}^k C_j (\nu_j - P_j)^2 . \quad (9.9)$$

The coefficients  $c_j$  are introduced because the deviations related to different values of  $p_j$  cannot be considered equal, since the deviation  $(\nu_j - p_j)$  of the same absolute value can be insignificant if the probability  $p_j$  itself is large, and very noticeable if it small. Therefore, the weights  $c_j$  must be taken inversely proportional to the probabilities  $p_j$ . Pearson proved that if you set:

$$c_j = n/p_j, j = \overline{1, k},$$

then for a large number of experiments  $n$  the distribution law of the value  $R$  has the following properties: it practically does not depend on the distribution law of a random variable  $\eta$  and only slightly depends on the number of experiments  $n$ , but depends only on the number  $k$  (the number of grouping intervals) and, as  $n$  increases, approaches the distribution  $\chi^2$  with density function:

$$f_k(r) = \left[ 2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right) \right]^{-1} r^{\frac{k}{2}-1} e^{-\frac{r}{2}} .$$

For such a choice of coefficients  $C_j$  the measure of discrepancy  $R$  is usually denoted by  $\aleph^2$

$$R = \aleph^2 = n \sum_{j=1}^k (v_j - P_j)^2 / P_j,$$

or, considering that  $v_j = n_j/n$ , where  $n_j$  – the number of sample elements in the  $j$ -th interval, we obtain

$$R = \aleph^2 = n \sum_{j=1}^k (n_j - nP_j)^2 / (nP_j). \quad (9.10)$$

The distribution  $\aleph^2$ , as it is known, depends on the parameter  $l$ , called the "number of degrees of freedom". Using Pearson's criterion, the number of degrees of freedom is assumed to be equal to the number of intervals  $k$  minus the number of independent conditions (constraints) imposed on the frequency  $v_j$ . Examples of such conditions can be  $\sum_{j=1}^k x_j v_j = m_x$  if there is a requirement of the statistical mean coincidence with a hypothetical mathematical expectation, or  $\sum_{j=1}^k v_j = 1$  if the sum of the frequencies was equal to 1 (this requirement is imposed in all cases), etc.

The principle of evaluating the result of the distribution law identification using the criterion  $\aleph^2$  is based on the theorem 9.1 (Pearson K.). Whatever the initial random variable  $\eta$  and the selected number of intervals  $k$  (such that  $p_j > 0, j = \overline{1, k}$ ) are, for each  $r > 0$  there is

$$\lim_{n \rightarrow \infty} P\{\aleph^2 \geq r\} = \int_r^{\infty} f_{k-1}(u) du.$$

A proof of this theorem can be found, for example, in [13].

Let  $\alpha$  – the value of such a level of significance that an event with probability  $\alpha$  is considered almost impossible. Then, solving the equation:

$$\int_r^{\infty} f_{k-1}(u) du = \alpha, \quad (9.11)$$

we find the value  $r = \aleph_{\alpha}^2$  corresponding to the significance level  $\alpha$  fixed by us. Then, under the condition  $R = \aleph^2 < \aleph_{\alpha}^2$ , the result of the distribution law identification of the random variable  $\eta$  is accepted, and if

$$R = \aleph^2 \geq \aleph_{\alpha}^2, \quad (9.12)$$

the result is rejected.

The indicated conclusion, obviously, depends on the chosen level of significance and therefore does not have an absolute character. More often than others, as  $\alpha$  there are taken the values 0,1 0.1; 0.05; 0.01; 0.005.

In practice, in order to determine the values of  $\mathcal{N}_\alpha^2$ , there are used tables in which solutions of the integral equation (9.11) for all possible values of  $\alpha$  and  $(k-1)$  are presented. These tables are provided in all textbooks on probability theory and mathematical statistics.

The presented scheme for evaluating the results of the distribution law identification by the Pearsons' criterion is presented in the form of the following algorithm [36]:

Step 1. Calculating the probabilities  $p_i, i=0, 1, \dots, k$  according to the hypothetical distribution law.

Step 2. Calculating the measure  $R = \mathcal{N}^2$ .

Step 3. Significance level value selection  $\alpha_m = \max\{\alpha_1, \alpha_2, \dots, \alpha_l\}$ .

Step 4. Verify the condition  $\alpha_m > 0$ . If this condition is violated, go to step 9.

Step 5. Define  $\mathcal{N}_\alpha^2 = R(l, \alpha)$ .

Step 6. Verifying the condition  $\mathcal{N}^2 < \mathcal{N}_\alpha^2$ . If this condition is met, go to step 8.

Step 7. Set  $\alpha_m = 0$ . Return to step 3.

Step 8. The output of the text "The selected distribution law is rejected".

Step 10. End.

### 9.5 Example of distribution law identification

Let consider the work of the address bureau kiosk located on the station square. As a result of observing the work of this kiosk for 500 business hours, the following statistics were obtained (Table 9.1):

Table 9.1 - Statistical data on the kiosk work

Number of requests received in 1 hour ( $x_i$ )	Request frequency, $n_i$	Relative frequency $v_j$
0	307	0,614
1	145	0,290
2	37	0,074
3	8	0,016
4	2	0,004
5	1	0,002
500		1,000

Using formulas (9.2) and (9.7), we determine the statistical estimates of the mathematical expectation and dispersion

$$\bar{x} = \frac{1}{n} \sum_{i=1}^k n_i x_i = \frac{256}{500} = 0.512;$$

$$S^2 = \frac{1}{1-n} \left( \sum n_i x_i^2 - n \bar{x}^2 \right) = 0.583$$

According to the table. 9.1 let create a histogram (Figure 9.1)

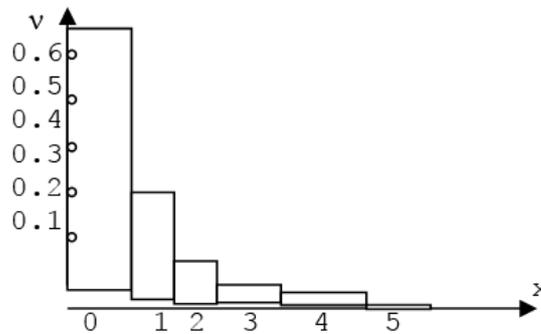


Figure 9.1 - Histogram

Comparing this histogram with graphs of the density functions of known theoretical distributions, we can see that it is similar to the Poisson distribution. However, it follows from formulas (4.5) that the values of mathematical expectation and dispersion should coincide for this distribution, which, unfortunately, does not have a place for the obtained estimates  $\bar{x}$  and  $S^2$ , since  $0.512 \neq 0.583$ .

This fact casts doubt on the correctness of the choice of the Poisson distribution. However, in a particular case, the data table. 9.1 satisfy all the basic properties of the Poisson law, such as the absence of aftereffect and the high probability of the occurrence of a zero number of events. Therefore, tentatively accepted:

$$\lambda = (0.512 + 0.583) / 2 = 0.5475,$$

we formulate the hypothesis that the experimentally obtained distribution can be considered as Poisson.

The validity of this hypothesis is verified by Pearson's criterion. The level of significance  $\alpha$  is equal to 0.05. In accordance with the algorithm given in Section 9.4, according to the Poisson formula (4.4), we determine the theoretical values of probabilities  $p_i$ :

$$p_0 = 0.577; \quad p_1 = 0.317; \quad p_2 = 0.087;$$

$$p_3 = 0.016; \quad p_4 = 0.002; \quad p_5 = 0.001.$$

Next, we calculate the value of the measure  $\aleph^2$  :

$$\aleph^2 = 3.52.$$

The next step of the algorithm is to find the critical value of measure  $\chi^2$  corresponding to the level of significance  $\alpha$  and the number of degrees of freedom  $l$ .

Let define the number  $l$ . As it is known, for the effective work of Pearsons' criterion it is necessary that, at grouping the possible values of the desired random variable, at least five experimental data should be included in each group. As can be seen from the table. 9.1, in the last two groups this condition is violated, therefore the number of groups of experimental data is reduced to four instead of six. The following theoretical probabilities will now correspond to these groups:

$$p_0=0.577; \quad p_1=0.317; \quad p_2=0.087; \quad p_3=0.016+0.002+0.001=0.019,$$

$$\sum_{i=1}^3 p_i = 1.$$

Then the number of degrees of freedom equals:

$$l=4-1-1=2,$$

since it is reduced by another one due to the fact that the  $\lambda$  value obtained from experimental data [34] was used to calculate the theoretical values of the probabilities.

The critical value of Pearson's criterion  $\chi_{\alpha}^2$  at the level of significance  $\alpha = 0.05$  and the number of degrees of freedom  $l = 2$  is 5.99 (see table 9.2). Consequently,  $\chi^2 < \chi_{\alpha}^2$  and the hypothesis that there are no significant differences between the empirically obtained distribution and the selected theoretical Poisson distribution with parameter  $\lambda = 0.5475$  is not rejected.

Table 9.2 - Experimental data

$\alpha$ 1	0.1	0.05	0.01	0.005
1	2.71	3.84	6.63	7.88
2	4.61	5.99	9.21	10.60
3	6.25	7.81	11.34	12.84
4	7.78	9.49	13.28	14.86
5	9.24	11.07	15.09	16.75
6	10.64	12.59	16.81	18.55
7	12.02	14.07	18.48	20.28
8	13.36	15.51	20.09	21.96
9	14.68	16.92	21.67	23.59
10	15.99	18.31	23.21	25.19
11	17.28	19.68	24.73	26.76
12	18.55	21.03	26.22	28.30
13	19.81	22.36	27.69	29.82

Continuation of 9.2 table

14	21.06	23.68	29.14	31.32
15	22.31	25.50	30.58	32.80
16	23.54	26.30	32.00	34.27
17	24.77	27.59	33.41	35.72
18	25.99	28.87	34.81	37.16
19	27.20	30.14	36.19	38.58
20	28.41	31.41	37.57	40.00

## Chapter 10. MASS SERVICE SYSTEMS MODELING

During the study of many real complex systems, it becomes necessary to solve problems related to mass service. However, the analytical apparatus of the mass service theory allows solving not all the problems of practical interest and is limited mainly to tasks in which the input flows are approximated by the simplest flow and the service time are assumed to be distributed according to the exponential law. In this regard, for the study of complex systems, reducible to the scheme of mass service systems (MSS), it is advisable to use computer modeling [5, 15].

The essence of this method in relation to the analysis of mass service systems is as follows. Using the previously described methods for random patterns modeling, there are formed implementations of a given request order flow. Next, the process of functioning of the service system is modeled. All system performance indicators of interest to the researcher are recorded. The general modeling algorithm repeatedly reproduces random implementations of the functioning process system under certain predetermined conditions. The information accumulated in this case is statistically processed.

We begin the study of computer modeling methods of MSS from the simplest case of systems with one serving channel.

### 10.1 Single-channel MSS modeling

Let the system receive requests that form an ordinary flow of homogeneous events with a given distribution law of intervals between events; requests are lined up and served in the order they were received. Considered MSS refers to mixed-type systems with a limited waiting time for requests in the queue. The waiting time is in most cases a random variable.

The duration of the requests service is also a random variable with a given distribution law.

The functioning of the MSS will be considered in the time interval  $[0, T_M]$ . This means that the request for which the start time of the service is  $t^h < T_M$ , and the end time of the service is  $t^{c^g} > T_M$ , is rejected.

As a result of the modeling, it is necessary to obtain the characteristics of the service quality: the share of served requests, average waiting time, and others.

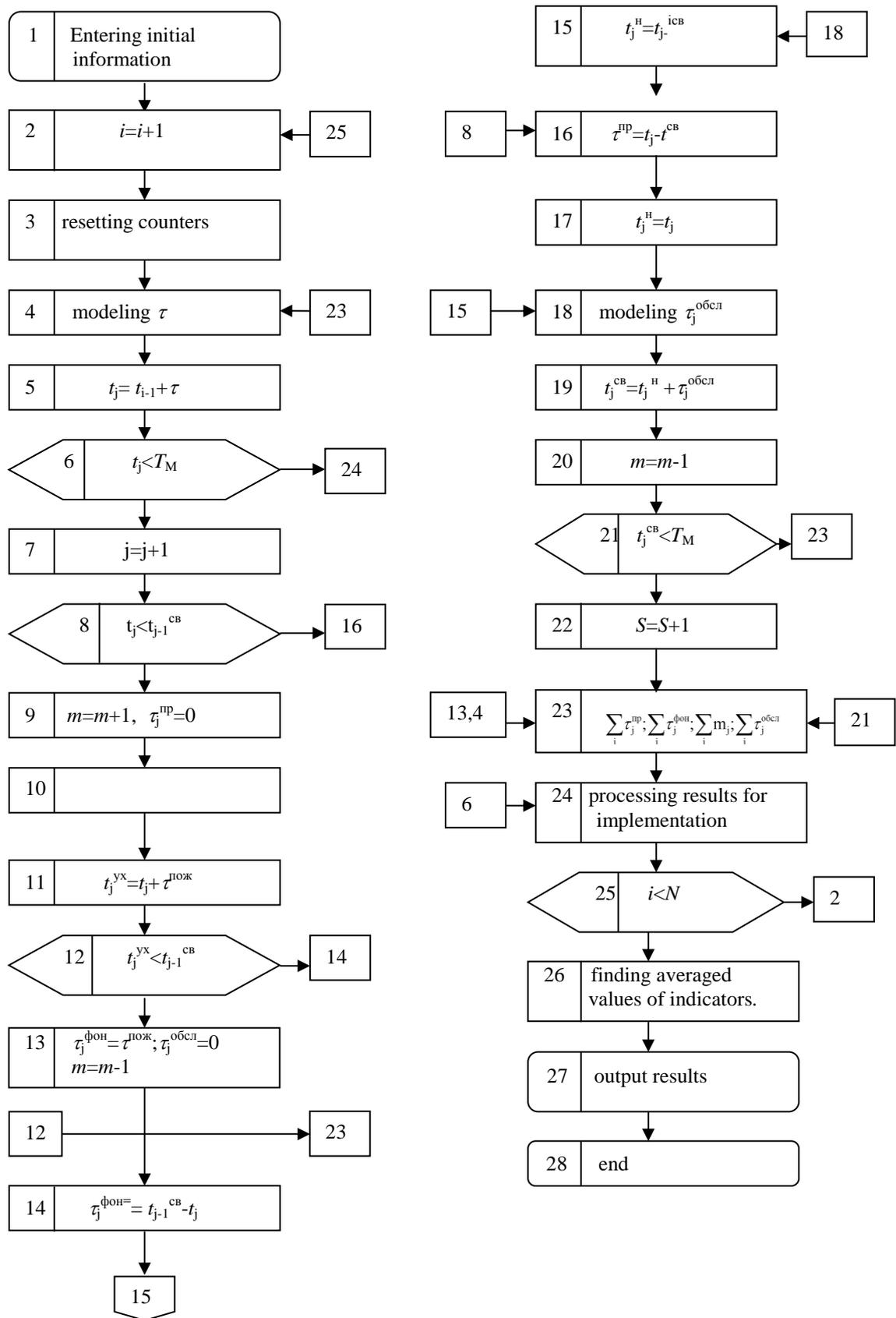


Figure 10.1 - Block diagram of single-channel MSS

The algorithm scheme simulating a given MSS is presented on the fig. 10.1. We describe its functioning. Operator 1 enters the input information (parameters of the requests flow, the distribution law of service time, etc.). Operator 2 is a counter of the number of implementations (runs) of the system functioning process. Operator 3 ensures the repetition of a new run by setting the initial conditions for the next implementation. Operators 4 and 5 form the moment of receipt of the next request  $t_j = t_{j-1} + \tau$ , where  $t_{j-1}$  – the moment of receipt of the previous request;  $\tau$  - time interval between these requests. Using operator 6, there is made a verification to see if the time of the request  $t_j$  arrives outside the modeling interval:

$$t_j < T_M. \quad (10.1)$$

If condition (10.1) is not satisfied, i.e. the modeling interval has been exhausted, you should proceed to the processing of modeling results, i.e. to the operator 24. If condition (10.1) is satisfied, then using operator 7, the number of requests received is calculated. Operator 8 finds out if the channel is busy or free at the moment of receipt of the  $j$ -th request. If the channel is busy at the time of receipt of the request, i.e.  $t_j < t_{j-1}^{ce}$ , then the control is transferred to the operator 9, which counts the current number of requests in the queue. Here  $m$  is the length of the queue. Operators 10 and 11 determine the estimated time the request leaves the queue. Here  $\tau^{noxc}$  - the waiting time limit for a request in the queue and is formed according to a given distribution law or is set in the form of a constant. Operator 12 checks the condition

$$t_j^{yx} < t_{j-1}^{ce}. \quad (10.2)$$

If this condition is satisfied, i.e. the waiting time of the  $j$ -th request exceeds the limit, then such a request, in accordance with the established discipline of waiting, must leave the system after  $\tau^{noxc}$ . For this purpose, control is transferred to the operator 13. For this request, the actual waiting time is equal to the limit, i.e.  $\tau^{\phi oxc} = \tau^{noxc}$ . Since the request leaves the system unserved, it is assigned a zero service duration  $t_j^{o\phi c1} = 0$ . The number of requests in the queue decreases by 1 ( $m = m-1$ ).

In this case, the analysis of the fate of the  $j$ -th request ends here and control from the operator 13 is transferred to the operator 23 for private data processing and subsequent modeling of the moment of the receipt of the next request, which will be discussed below.

If condition (10.2) is violated, then the request does not leave the system and control from the operator 12 is transferred to the operator 14 to calculate the actual waiting time of the request in the queue, which is equal in this case

$$\tau^{\phi oxc} = t_{j-1}^{ce} - t_j. \quad (10.3)$$

Operator 15 generates the moment of the service start, coinciding with the moment of release of the channel, i.e.  $t_j^H = t_{j-1}^{c6}$ .

Let return to the operator 8. If the channel is free at the time of receipt of the request ( $t_j \geq t_{j-1}^{c6}$ ), then this request is accepted for service and it is possible to form (operator 18) the service duration of the request  $\tau_j^{o6c7}$  as a random variable obeying the corresponding distribution law. However, it is firstly necessary to determine the idle time of the channel (operator 16) and to form (operator 17) the moment the service starts, which is equal in this case to the moment the request enters the system, i.e.

$$t_j^H = t_j.$$

Operator 19 determines the moment of channel release after the  $j$ -th request is finished  $t_j^{c6} = t_j^H + \tau_j$ , and operator 20 reduces the number of requests in queue by 1. However, the  $j$ -th request is considered served and fixed by the operator 22, if the condition is satisfied:

$$t_j^{c6} < T_M, \quad (10.4)$$

verified by the operator 21.

In case of violation of inequality (10.4), this request is rejected and control is transferred to the operator 23 for private data processing.

Private data processing consists in the preparation of certain sums necessary for the subsequent statistical calculation of mathematical expectations and the probabilities of certain values selected as indicators of MSS effectiveness.

Sums are:

$\sum \tau_j^{\phi o \omega c}$  - total duration of the actual waiting time for all requests received in

the system;

$\sum \tau_j^{np}$  - total channel idle duration time for the system operation interval  $[0, T_M]$ ;

$\sum \tau_j^{o6c}$  - total duration of the service channel;

$\sum m_j$  - the total sum of queue lengths at the time of receipt of requests.

Let return to the operator 6. If condition (1) is not satisfied, i.e. the moment the next request arrives is outside the modeling interval, it is necessary to proceed to processing the results for the  $i$ -th implementation of the simulation model. In this case, control from the operator 6 is transferred to the operator 24, in which the estimates of mathematical expectations and the calculation of probabilities are calculated.

The constructed modeling algorithm allows to obtain the following indicators:

– average length of stay of the request in queue:

$$\bar{\tau}^{фo\omega\kappa} = \frac{1}{j} \sum \tau_j^{фo\omega\kappa};$$

- average queue length:

$$\bar{m} = \frac{1}{j} \sum m_j;$$

- probability of serving an request:

$$p_{o6c\lambda} = S/J;$$

- probability of denial of service:

$$p_{om\kappa} = 1 - S/J;$$

- average channel idle time:

$$\bar{\tau}^{np} = \frac{1}{j} \sum \tau_j^{np};$$

- probability of channel idle:

$$p_{npocm} = \tau_{npocm}/T_M.$$

As it is known, in order to achieve a given accuracy of the obtained estimates, it is necessary to average the results of a large number of implementations. Operator 25 provides the execution of a given number ( $N$ ) of model implementations, and operator 26 calculates the averaged values of MSS performance indicators.

## 10.2 MSS modeling with unreliable elements

During the study of many real systems that are reduced to mass service schemes, it is necessary to take into account the possibility of failure of service channels. Let consider this addition applied to the single-channel MSS model so that before the acceptance of the next service request, the process physicist remains the same (Section 10.1). We assume that the channel has a certain uptime, which is a random variable with a given distribution law. If the channel fails at the moment when the request is being serviced, it is rejected.

We will create a sub-algorithm for counting the unreliability of the service channel (Figure 10.3). The sub-algorithm is included after the operator 21 of the main modeling algorithm (Figure 10.1) using the  $R$  attribute:

$R=1$ , if the model takes into account channel unreliability;

$R=0$ , otherwise.

This feature is taken into account by the operator 21.1, the initial operator of the sub-algorithm. Operator 21.1 verifies the condition

$R=1.$

(10.5)

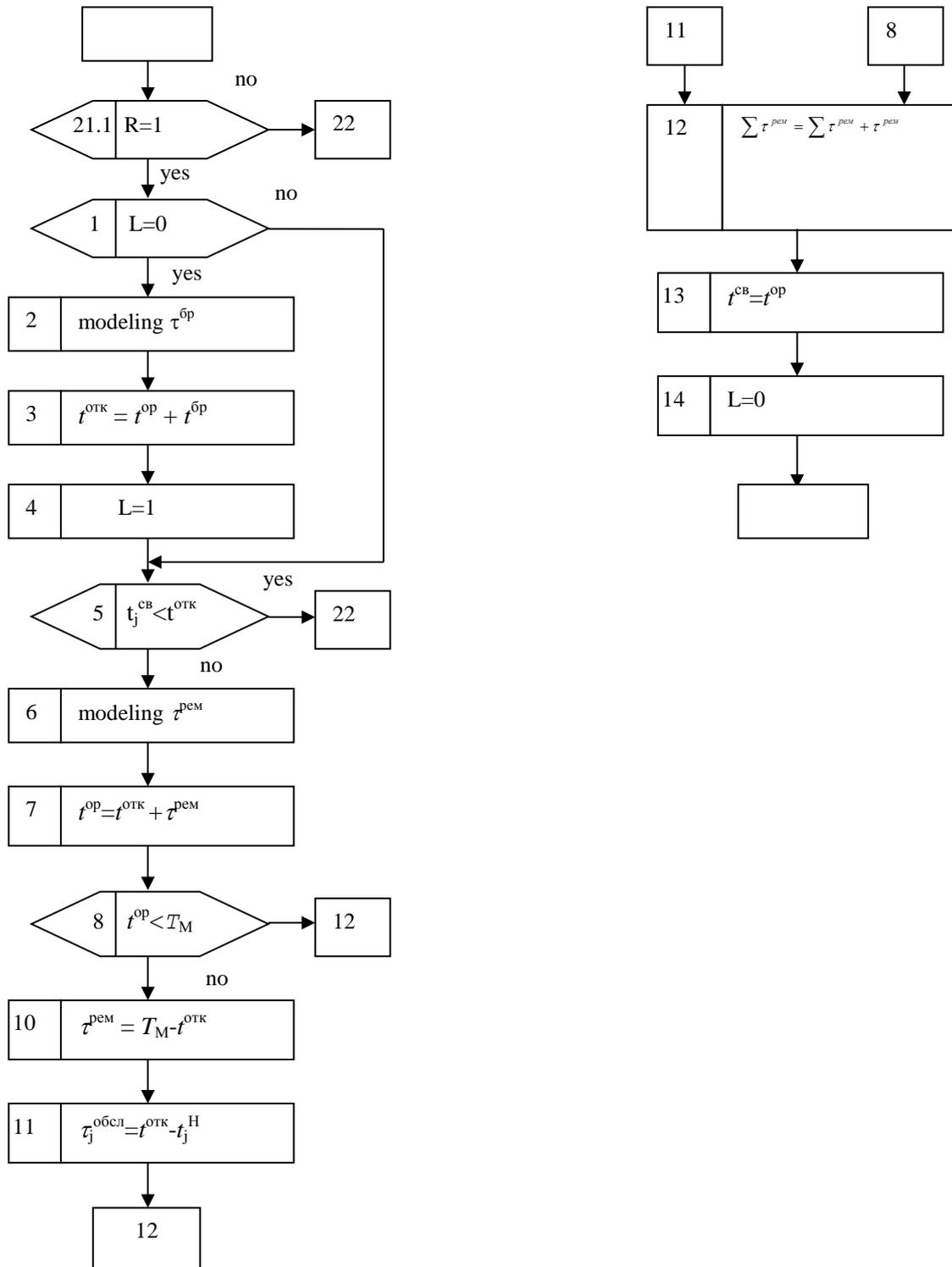


Figure 10.2 - Service channel unreliability counting sub-algorithm

If the condition (10.5) is not satisfied, i.e. channel unreliability is not taken into account, control is transferred to the operator 22 of the main algorithm, and process modeling is performed, as was shown on the figure 10.1. When this condition is met, MSS modeling is carried out taking into account failure and

restoration of channels, for which control is transferred to the operator 1 (Figure 10.2) in order to verify the condition

$$L=0. \quad (10.6)$$

The essence of this condition is as follows. Modeling the state of the channel for each current point in time, it is necessary to know the nearest subsequent moment of channel failure  $t^{omk}$ .

For the timely development of the moment  $t^{otk}$  of the channel there is introduced the attribute  $L$ :

- $L=1$ , if the nearest subsequent channel failure moment is calculated;
- $L=0$ , if the moment of failure has exhausted itself and it is necessary to determine the next similar moment.

Thus, if condition (10.6) is satisfied, then the group of operators 2, 3, 4 calculates the closest moment of failure and accordingly changes the value of the attribute  $L$ . If this condition is not met, control from the operator 1 is immediately transferred to the operator 5, in which the most important verification for this scheme - whether the channel fails during the service period of the  $j$ -th request, i.e. the condition is verifying:

$$t_j^{cs} < t^{omk}. \quad (10.7)$$

If the condition (10.7) is satisfied, i.e. when the  $j$ -th request is finished servicing before the channel's failure (which does not affect the fate of the request), the control is transferred to the operator 22 of the main scheme (Fig. 10.1).

If this condition is not met, then at the time of failure of the channel, the  $j$ -th request is under maintenance and, therefore, is rejected, and control is transferred to the operator 6 to work out the repair duration and to determine (operator 7) the channel recovery time. Next, the operator 8 verifies whether the repair will end outside the modeling interval, i.e.

$$t^{op} < T_M. \quad (10.8)$$

If the condition (10.8) is not satisfied, then after determining a number of parameters (operators 9, 10, 11), control is transferred to the operator 12. This operator, for any outcome of condition (10.8), calculates the total repair duration

$$\tau_s^{pem} = \tau_s^{pem} + \tau^{pem}.$$

The operator 13 records the moment of completion of the repair of the channel as the moment of its release, i.e.  $t^{ce} = t^{op}$ . Since the next moment of the channel failure has exhausted itself, the operator 14 assigns a zero value to the attribute  $L$ . This completes the operations associated with the failure and restoration of the

channels, and control is transferred to the operator 23 of the main algorithm (Fig. 10.1).

### 10.3 MSS modeling with relative priority of service

Let consider a multi-channel MSS, the input of which, unlike previous systems, receives not one, but  $Q$ , ( $q=1,2,\dots,Q$ ) request flows, each with its own distribution law. Moreover, the flows are unequal in importance. So, a request from a  $q$ -th flow is more important than a request from a  $(q+1)$ -th flow, and the latter, in turn, is more important than a request from a  $(q+2)$ -th flow, etc. All requests of one flow are equivalent.

The system serves requests taking into account the entered priority, i.e. a request with the lowest priority number from the queue always arrives in the freed channel. If the service has a request of a lower priority, then its service continues to the end even if during the process of its service a request of a higher priority enters the system, i.e. relative priority service takes place.

The structure of the modeling algorithm for MSS with a relative priority of service has significant differences from the previously considered systems. This is due to the need to simulate several incoming request flows and form several queues according to the number of priorities, as well as the difficulty of choosing from these queues the requests for servicing. The processing of modeling results also has differences, since most indicators are calculated separately for each of the priorities.

We create a modeling algorithm for such a MSS. The scheme of the modeling algorithm begins with the operator 1 (Fig. 10.3), which inputs the initial information: the modeling interval  $T_m$ , the number of service channels -  $k$ ; the number of request flows -  $Q$ , as well as the parameters of the distribution laws of request flows, service time, etc. For each  $k$ -th channel, the initial value of the moment of release of the service channel  $t_k^{cs}=0$  is also set. Naturally, the initial value of the minimum moment of release of the service channel is also equal to zero ( $t^{cs.M}=0$ ).

Before we begin the description of the work of the operators modeling the input request flows, we briefly dwell on the methods of queuing. In the initial period of system operation queues must be formed according to the method of "one request of each priority" [15]. Further, it is more expedient to switch to another method, which can be characterized as the method of "filling empty space" [37].

The choice of one of the two methods of queuing is determined by the value of the attribute  $L$ :

$L=1$ , if the queue is formed according to the "one request of each priority" method;

$L=0$ , if queuing is formed using a different method.

At the initial moment  $L=1$ .

Modeling of the request flow begins with the highest priority (operator 2). Operator 3, on the basis of a given distribution law, forms the interval between

neighboring request of  $q$ -th priority, and operator 4 determines the moment of receipt of the first request of the  $q$ -th priority.

Operators 5, 6, 7 verify for completion of the modeling process for this implementation. The modeling process is considered complete when, for all  $Q$  incoming flows, the condition of the operator 5 is satisfied:

$$t_{q1} \geq T_M. \quad (10.9)$$

The number of flows for which the modeling interval has been exhausted is calculated by the counter 6. And finally, if the condition is satisfied (operator 7):

$$M \geq Q, \quad (10.10)$$

that means the end of the modeling process, control is transferred to the operators of the processing of the results. If condition (10.10) is not fulfilled, control is transferred to the operator 12, from which the modeling of the process of selecting request from the queue begins.

We return to the operator 5, to the case when condition 10.9 is not fulfilled (the moment the request arrives falls in the modeling interval). In this case control is transferred to the operator 8, counting:

- the number of requests of  $q$ -th priority received by the system ( $j_q=j_q+1$ );
- total number of requests received by the system ( $j=j+1$ );
- the number of requests of the  $q$ -th priority in the queue ( $m_q=m_q+1$ ).

Further, when the conditions of operators 9 and 10 are met with the help of the operator 11, a queue is formed according to the method of “one request of each priority”. And if the conditions of operators 9 and 10 are not fulfilled, control is transferred to the operator 13. In this case, after the operator 10, the value of the attribute  $L$  changes, since further during the formation of a request queue only the "filling empty space" method will be used.

From the operator 13, the next stage of modeling begins - the selection of requests from the queue for servicing. In accordance with the discipline of service adopted in this model, it is necessary to turn to the high priority queue and select from it the request that came before everyone else. Therefore, this operation begins with the first priority queue (operator 13). Then there is verified (operator 14) the presence of a queue of  $q$ -th priority according to the condition:

$$m_q > 0. \quad (10.11)$$

If condition (10.11) is not satisfied, i.e. there are no requests of  $q$ -th priority in the queue, operators 15 and 16 call the queue of the next highest priority. The cycle is repeated in this way until the presence of requests in the queue of any priority will be revealed, i.e. condition (10.11) will be satisfied. In this case, control

from the operator 14 is transferred to the operator 17, and for the first order of the  $q$ -th queue, the condition is verified:

$$t_{q1} \leq t^{CB.M} \quad (10.12)$$

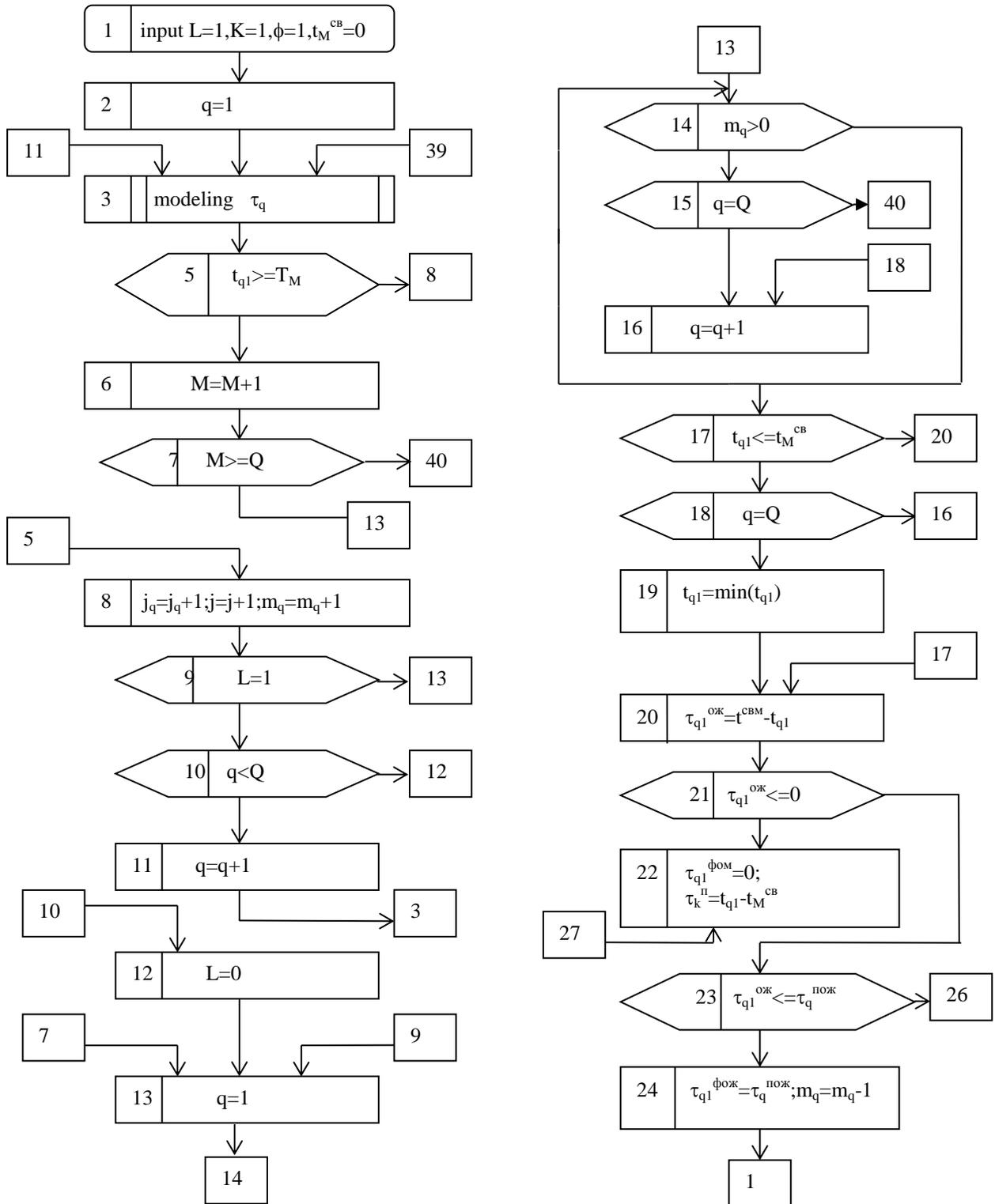


Figure 10.3 - MSS algorithm with relative priority

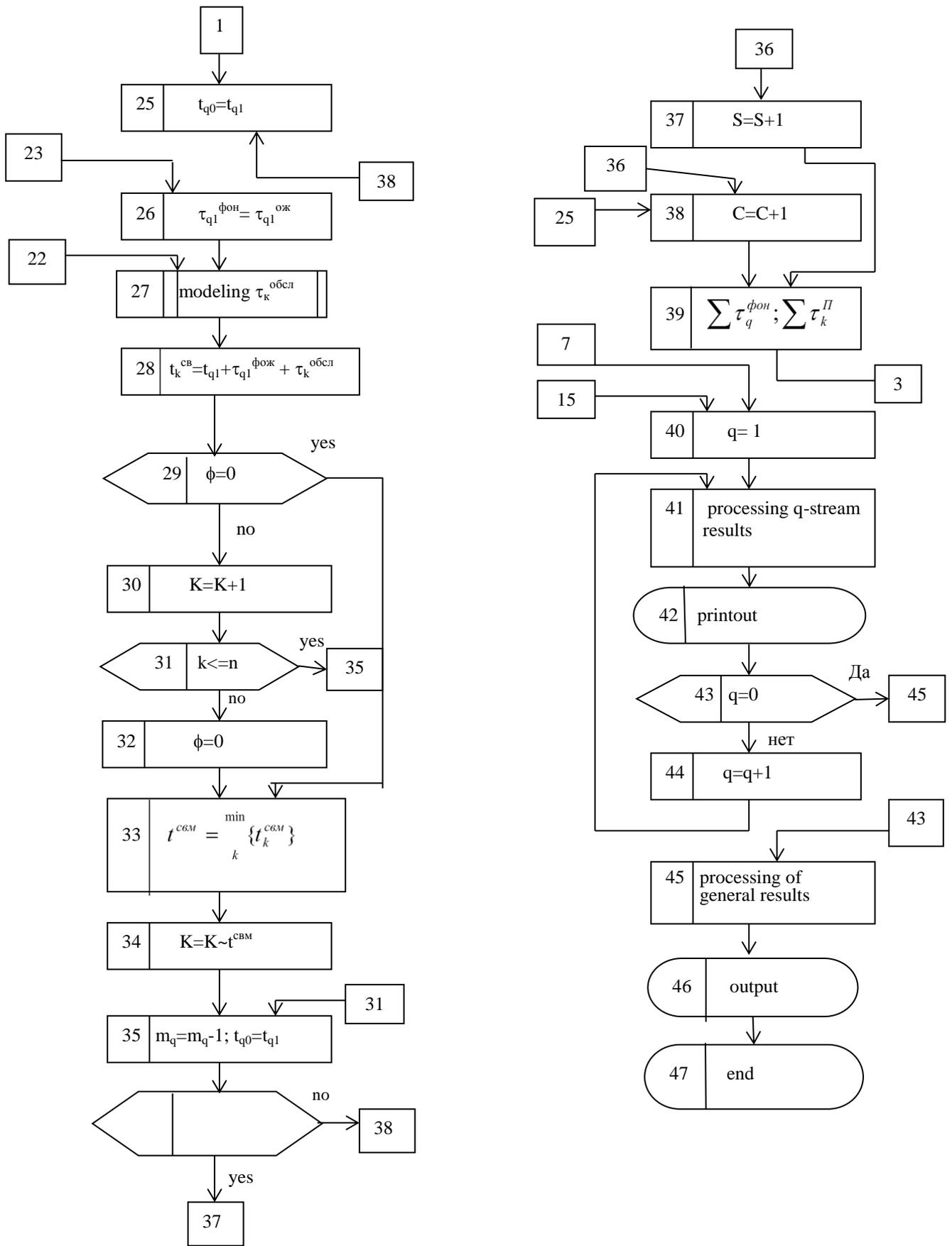


Figure 10.3 - Continuation of the algorithm

If this condition is satisfied, i.e. by the time of the release of at least one of the channels ( $t^{c6.M}$ ) at the moment of time  $t_{q1}$  when a request has been received, then it is sent for servicing (to the operator 20). Otherwise, i.e. if at the moment of time of channel release there are no requests in the  $q$ -th queue, you should check if there are any requests in the small priority queues. For this, control from the operator 17 is transferred to the operator 8, and if the condition  $q=Q$  is not fulfilled, then control is transferred to the operator 16 in order to call requests of the next highest priority. The chain of operators 14, 15, 16, 17, 18 will operate until an request satisfying condition (10.12) is found in any queue, or until after checking all the queues, no such request appears in any of them. In the last case, control from the operator 18 is transferred to the operator 19 in order to select the first-time request for servicing, regardless the priority  $t_{q1} = \min\{t_{q1}\}$ .

Request selected in this way arrives for servicing, which is implemented by transferring control to the operator 20. Operator 20 calculates the waiting time for the request in the queue, and operator 21 determines whether the request really expected the channel release or if there were simple service channels. In the last case, the operator 22 calculates the idle time of the service channel. If the request was waiting, then control from the operator 21 is transferred to the operator 23 in order to verify the condition:

$$\tau_{q1}^{o\omega\kappa} \leq \tau_q^{no\omega\kappa}. \quad (10.14)$$

Here  $\tau_q^{no\omega\kappa}$  – the maximum possible waiting time for the request service start of  $q$ -priority. If the condition (10.14) is violated, this request leaves the queue and, therefore, the actual time for request waiting in the queue  $\tau_{q1}^{\phi o\omega\kappa}$  equals the maximum permissible, and the number of requests in this queue decreases by one (operator 24). Next, control through the operator 25 is transferred to the failure counter (operator 38).

When condition (10.14) is met, control is transferred to the operator 26 in order to fix the actual waiting time of the request in the queue. The operator 27 models the duration of the request service by the channel, and the operator 28 determines the moment of release of the  $k$ -th channel after request service.

The next group of operators from 29 to 32 provides a consistent involvement to the service of all channels at the initial moment of the system operation. This group of operators begins from the verification of the value of the attribute  $\psi$ , determined from the condition:

$\psi=1$ , if not all channels have been involved to the service yet;

$\psi=0$ , if all channels have been released at least once.

When the conditions of the operator 19 are fulfilled, the channels are involved in request service in increasing order of the channel number (operator 30) until all reserves have been exhausted (operator 31). After this, the attribute  $\psi$  is assigned a zero value (operator 32).

If the condition of the operator 29 is not fulfilled, then the operator 33 determines the moment of readiness for work  $t^{c6.M}$  of the first of the released channels, and the operator 34 remembers the number of this channel as a candidate for servicing the next request. The operator 35 imitates the shift of the queue forward by one request accepted for service. However, this request will be served and recorded by the counter of served requests (operator 37) only if the conditions of the operator 36 are fulfilled, i.e. if the moment of completion of the request service falls within the modeling interval  $[0, T_M]$ . If the condition  $t_K^{c6} \leq T_M$  is violated, the request leaves the system without a service, which is fixed by the failure counter (operator 38). After both counters, control by the operator 39, in which the total values of channel idle time and request waiting time are calculated, is transferred to the operator 3 in order to simulate the next request in the considered queue. The last group of operators (40-47) processes the modeling results firstly for each request flow (operators 40, 41, 42, 43), and then for the system as a whole. The described modeling process for multi-priority MSS continues until the modeling interval has been exhausted for all flows.

## Chapter 11. PRODUCTION SYSTEMS MODELING

### 11.1 Value chains

The main purpose of this section is to study and simulate the dynamics of the functioning of value chains (VC).

Let consider the general scheme of the object of study, including production and sales with its warehouses (Figure 11.1).

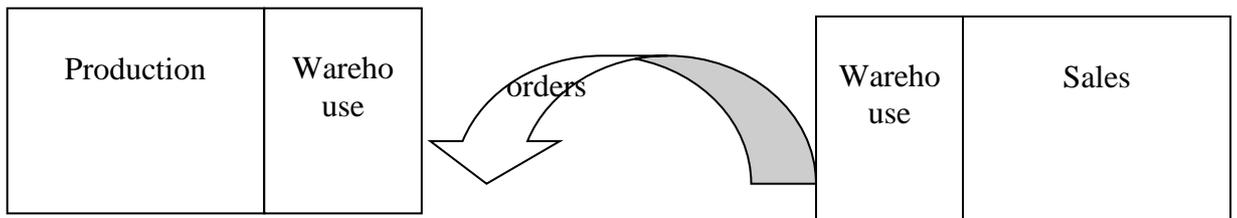


Figure 11.1 – Object of study

The description and formalization of the object of study will begin with the sales link, i.e. production will play a passive role of the wholesale base of the supplier of goods. The figure 11.2 shows the functioning of the object of study, which is designed to analyze the influence of individual parameters of the system on its effectiveness.

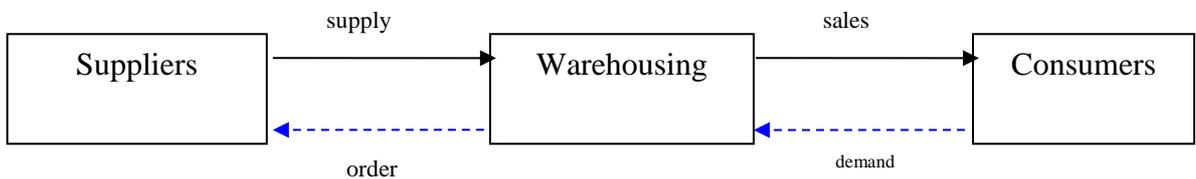


Figure 11.2 – The scheme of functioning of the object of study

In the block "Consumers" a random flow of orders  $\{t, D\}$  is formed, where  $t$  - the moments of orders receipt, and  $D$  - the value of demand. In the blocks "Warehousing" and "Suppliers", respectively, storage and organization of deliveries to the warehouse are carried out. The main parameters of this system are the quantity (random) of demand  $D$ , the critical value of the goods at the warehouse  $V^{cr}$ , the strategy for determining the value of supplies from the wholesale base.

We will create a technological scheme of the functioning of the object of study on the basis of the principle of "flowing tank" (Fig. 11.3). This principle was firstly used by J. Forrester [52] and involves the alternation of blocks characterizing volume indicators with blocks characterizing flow intensities. Indeed, the volumes are obtained as the difference between the input and output intensities of the studied parameter.

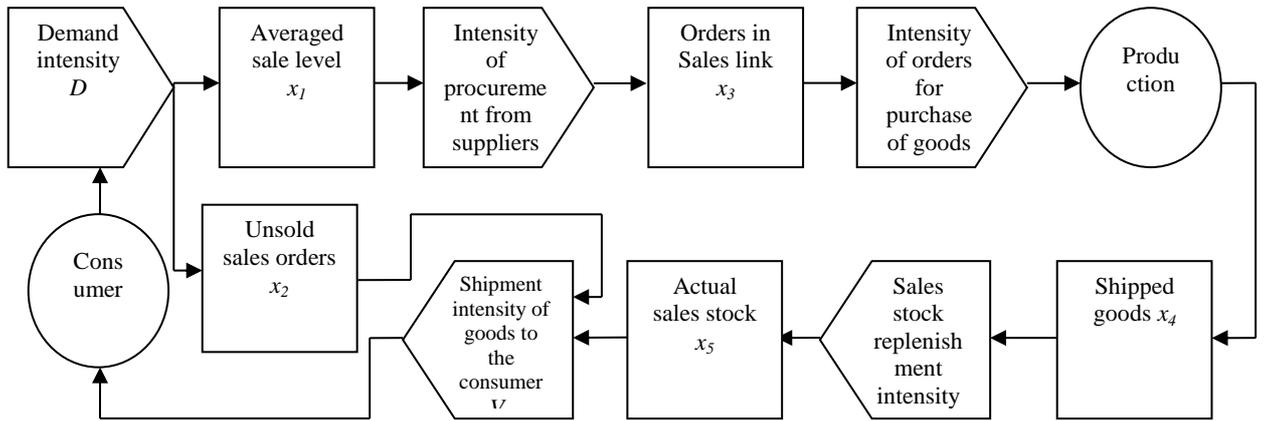


Figure 11.3 – Technological scheme of the second module

The main functions of the production link are clearly shown on the figure 11.4. the presence of an internal production warehouse is assumed. This functional structure is characteristic for PMS, in which production volumes are quite strongly correlated with sales, for example, in the food industry, where the shelf life of goods plays a significant role [40, 41].

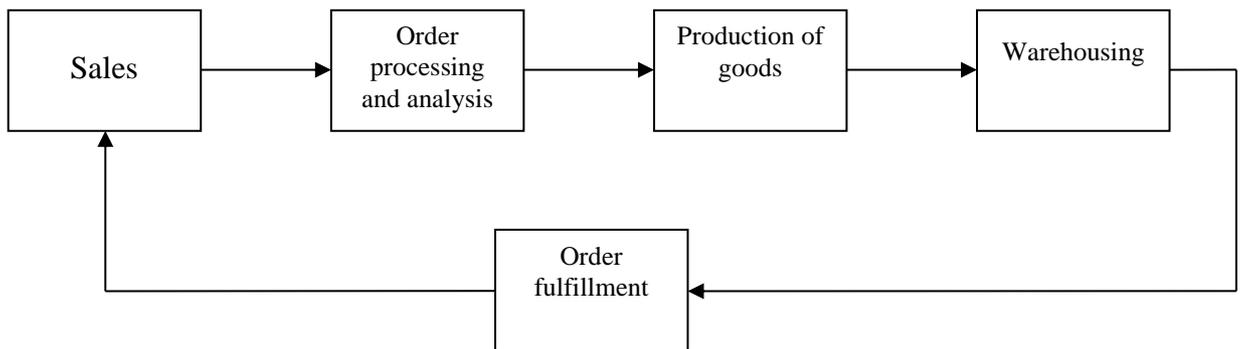


Figure 11.4 – Functional structure of the first module

The following functional structure of the VC production link (Fig. 11.5) is more universal. Here, the “Stocks” block plays a rather independent role not only as a factory warehouse, but to some extent as a wholesale base.

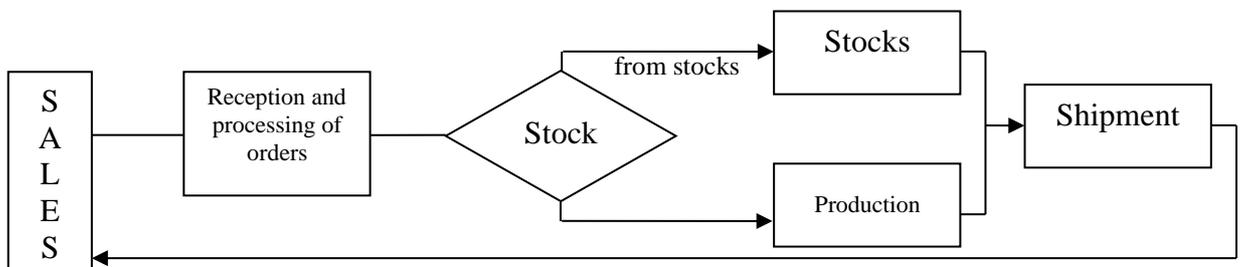


Figure 11.5 – Functional structure of the second module

All the main functions of the production link are also clearly displayed in the functional structure. We will only explain the purpose of the block "Order analysis". Depending on the volume of actual stocks of goods at the warehouse and some other production parameters, there are determined the relationships between the intensities of order fulfillment from production stocks and due to products manufactured at the factory.

We will construct the following technological scheme of the VC functioning (Figure 11.6) based on the functional structure of the first module.

Based on the principle of "flowing tank" [52], the stages of the technological process corresponding to the intensities of material and information flows alternate with the stages of the formation of volumetric process parameters. Therefore, the volume values (shown on the figure in the form of a rectangle) are calculated as the difference between the input and output intensities of the blocks.

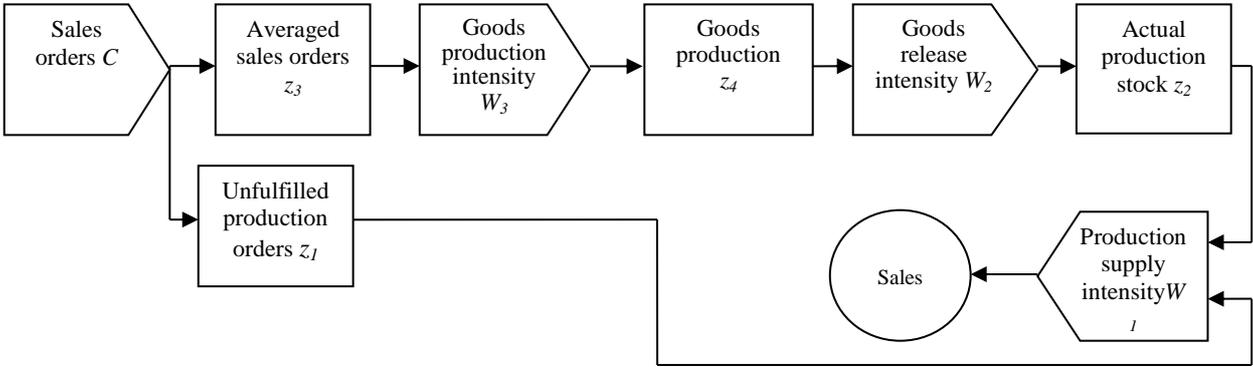


Figure 11.6 – Technological scheme of the first module

The intensity of sales orders  $C$  is, depending on the type of product, a continuous or integer random variable. The average value of sales orders  $z_3$  is an important indicator for planning the production intensity  $W_3$ . The manufacturing process and the intensity of finished products are displayed by parameters  $z_4$  and  $W_2$ . Following the principle of "flowing tank", the parameters  $W_2$  and  $W_1$  are the input and output of the actual stock block  $z_2$  at the production warehouse. The intensity of the goods supply  $W_1$  for sales also depends on the volume of completed orders  $z_1$ .

We proceed to the construction of the technological scheme of the second module functioning (Figure 11.7). Here we also use the principle of "flowing tank".

A significant difference between this scheme and the first ones is not only the above expansion of the role of the factory warehouse, but also the consideration of the factor of ensuring production with appropriate raw materials.

At the initial stages of the scheme, characterized by the parameters  $d_1, f_2, f_3$ , incoming orders are processed and decisions are made on the ratios of the order fulfillment intensities from stocks and due to production. The following two parallel chains of blocks, defined, respectively, by the parameters  $d_7, f_7$  and  $d_2, f_4$  complete the process of fulfilling orders both from stocks and from production.

Parameter  $d_4$ , corresponding to the value of the average shipment of products from stocks, is the basis for determining the volume of products needed to reimburse stocks. From this block begins the technological scheme chain, characterized by the parameters  $f_6, d_6, f_5, d_3$  and responsible for the compensation of products stocks at the production warehouse. Another chain of the technological scheme, starting with the block “Averaged order intensity” and characterized by the parameters  $d_5, f_9, d_9, f_8, d_8$ , refers to the processes of providing production with the necessary raw materials.

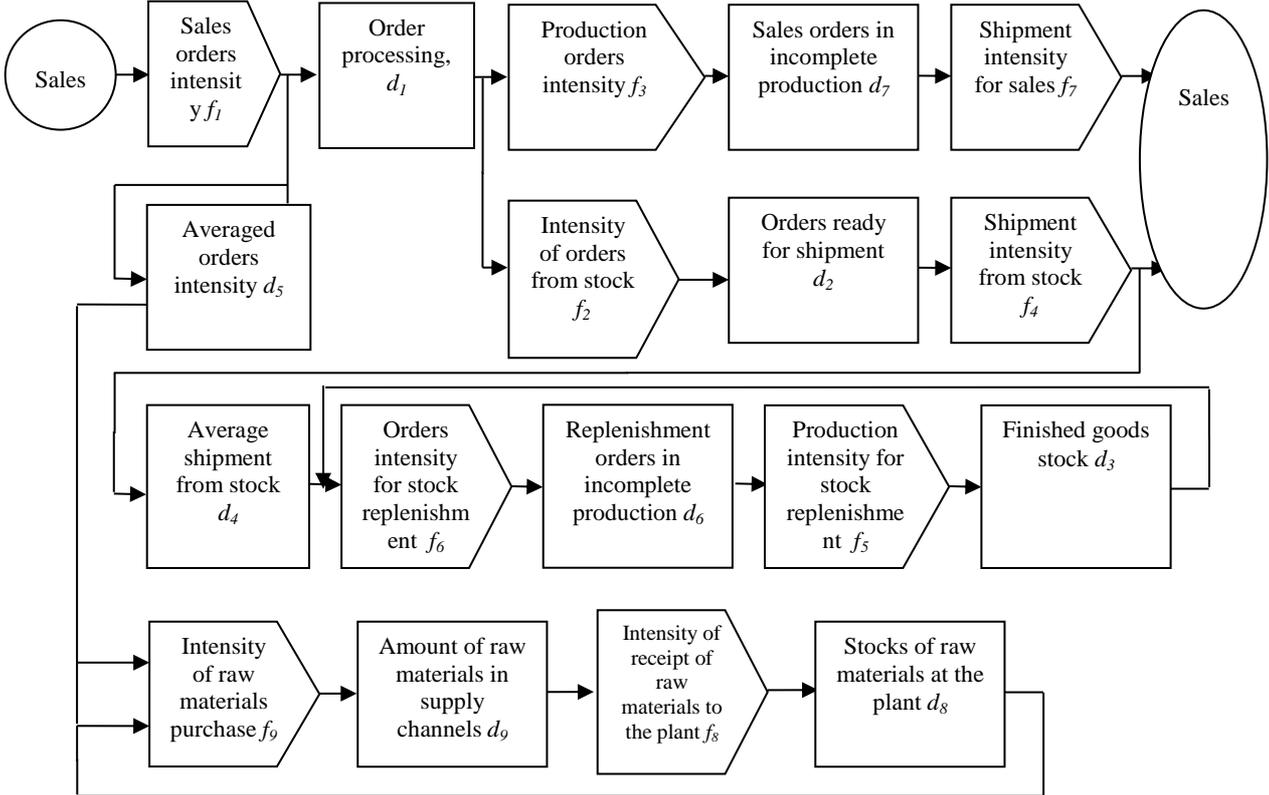


Figure 11.7 – Technological scheme of the second module

**11.2 Development of a model for the VC functioning**

We will also begin the development of the VC model by creating the “Sales” link model.

The designations used in this study are given below:

- $t_i$  - previous moment of time;
- $t_j$  - current moment of time;
- $t_k$  - future moment of time;
- $\tau$  - time interval between solutions of equations;
- $a_1$  - constant, which is a fraction of the difference by which it is necessary to adjust the sales intensity of goods;
- $a_2$  - order delay in sales link;

- $a_3$  - coefficient of proportionality between the average level of sales and the desired sales stock;
- $a_4$  - delay in transporting goods to the sales link;
- $a_5$  - minimum delay in order fulfillment by the sales link;
- $a_6$  - delay in order fulfillment of organizational character.

Let describe the equations of processes occurring in the technological scheme shown on the figure 11.5. Since the external demand flow is formed in the “demand intensity” block, we begin by describing the processes from the “averaged sales level” block. The averaged level of sales is an important parameter of the system on which the plans for the formation of stocks of goods (products) at the warehouse and the volume of orders to replenish these stocks are based. Averaging, which is based on the dependence described by the exponential function, gives the most significant result in case when it covers data obtained last of all. Let consider two approaches in order to find it. The first approach is associated with exponential averaging [52]. Here, the average sales value  $x_1[t_j]$  at the current moment of time  $t_j$  is obtained as the previous value of averaged sales at time  $t_i$ , adjusted for a fraction of the difference between the demand intensity during the last time interval  $D[t_j-t_i]$  and the previously calculated average value of sales  $x_1[t_i]$ :

$$x_1[t_j] = x_1[t_i] + \tau \left( \frac{1}{a_1} \right) * (D[t_j - t_i] - x_1[t_i]), \quad (11.1)$$

The second approach to determining averaged levels of sales involves counting for product prices. Let  $P_i$  – the product price in the previous time interval, then:

$$x_1[t_j] = C + RP_i + W \quad (11.2)$$

Coefficients  $C$  and  $R$  are determined by standard econometric methods.  $W$  is a random variable characterizing various factors associated with production technology.

Here it is assumed that the forecast of the averaged sales level for the new interval starting from the moment of time  $t_j$  is carried out on the prices of the previous interval  $P_i$ . Therefore, considering the more realistic approach, in the previous formula it is necessary to use the expected product price [19]. This price takes into account the previous trend in price changes and is determined by the formula:

$$P_j = P_i + \beta P_{i-1},$$

where  $\beta$  – parameter characterizing the weight fluctuations of prices ( $0 \leq \beta \leq 1$ ). Then the expression for the averaged level of sales will take the form:

$$x_1[t_j] = C + R(P_i + \beta P_{i-1}) + W. \quad (11.3)$$

The intensity of purchase from suppliers  $V_1[t_k-t_j]$  from the moment of time  $t_j$  to time  $t_k$  in a subsequent interval is determined by the following equation:

$$V_1[t_k - t_j] = D[t_j - t_i] + \frac{1}{a_2} \{ (a_3 x_1[t_j] - x_5[t_i]) \}, \quad (11.4)$$

The meaning of this equation is also quite transparent. Obviously, the intensity of retail purchases will primarily depend on the prevailing demand intensity  $D[t_j-t_i]$  to replenish the stock in exchange for the goods sold. The second part of the equation is a correction of the existing intensity depending on the difference between the desired  $a_3 x_1[t_j]$  and the actual  $x_5[t_i]$  stock, taking into account the delay  $a_2$ .

The value of orders of the sales link  $x_3[t_j]$ , which are at the design stage, is determined from the equation:

$$x_3[t_j] = x_3[t_i] + \tau (V_1[t_j - t_i] - V_2[t_j - t_i]) \quad (11.5)$$

In accordance with the principle of "flowing tank" the variable  $V_2[t_k-t_j]$  is the intensity of goods purchase issued by the sales link and is determined by the equation:

$$V_2[t_k - t_j] = x_3[t_j] / (a_2), \quad (11.6)$$

Deliveries of goods to the sales link, reflected in the blocks "shipped goods" and "replenishment intensity", are described by equations (11.7) and (11.8) and have the following structure similar to (11.5) and (11.6):

$$x_4[t_j] = x_4[t_i] + \tau (W_1[t_j - t_i] - V_3[t_j - t_i]), \quad (11.7)$$

$$V_3[t_k - t_j] = x_4[t_j] / (a_4) \quad (11.8)$$

Here  $V_3$  – deliveries received by the sales link;  $W_1$  - deliveries from stock. Similarly, you can get an equation that determines the actual sales stock:

$$x_5[t_j] = x_5[t_i] + \tau (V_3[t_j - t_i] - V_4[t_j - t_i]) \quad (11.9)$$

and the value of unfilled orders by the sales:

$$x_2[t_j] = x_2[t_i] + \tau (D[t_j - t_i] - V_4[t_j - t_i]) \quad (11.10)$$

For the intensity of shipment of goods to consumers  $V_4[t_k-t_j]$  we get the expression:

$$V_4[t_k-t_j]=\begin{cases} \frac{x_2[t_j]}{(a_5+a_6*a_3*x_1[t_j]/x_2[t_j])} \leq \frac{x_5[t_j]}{\tau} \\ \frac{x_5[t_j]}{\tau} < \frac{x_2[t_j]}{(a_5+a_6*a_3*x_1[t_j]/x_2[t_j])} \end{cases}, \quad (11.11)$$

Intensity value  $V_4[t_k-t_j]$  equals to the left side of the inequality that is true in a particular case.

Let proceed to the development of a model of the "production" process control system.

The designations used in this study are given below:

- $z_1$  - the amount of unfulfilled production orders;
- $z_2$  - actual stock;
- $z_3$  - average value of sales orders;
- $z_4$  - volume of unfinished production;
- $C$  - sales order intensity;
- $W_1$  - intensity of product supply to sales link;
- $W_2$  - product intensity at the output;
- $W_3$  - production intensity at the input;
- $f_1$  - the intensity of receipt of orders for the plant (unit of time, hereinafter - unit of time);
- $f_2$  - the intensity of orders fulfilled due to stocks of the plant (unit of time);
- $f_3$  - intensity of orders satisfied by production (unit of time);
- $f_4$  - the intensity of shipment of products from stocks of the plant (unit of time);
- $f_5$  - production intensity for stock replenishment at the plant (unit of time);
- $f_6$  - stock replenishment order intensity;
- $f_7$  - sales order shipping intensity;
- $f_8$  - the intensity of receipt of raw materials at the plant;
- $f_9$  - raw material purchase intensity;
- $d_1$  - volume of orders in the process of registration at the plant;
- $d_2$  - orders at the expense of stocks that are already issued but not yet completed;
- $d_3$  - stock of finished products in production;
- $d_4$  - average value of products shipment from production stocks;
- $d_5$  - averaged order intensity;
- $d_6$  - replenishment orders at the manufacturing stage;
- $d_7$  - orders in incomplete production;
- $d_8$  - stocks of raw materials in production;

$d_9$  - the amount of raw materials in the supply channels.

Let start with a description of the equations of processes occurring in the technological scheme shown on the figure 11.8. Firstly, we determine the volumetric process parameters at fixed moments of time. The averaged value of sales orders is an important parameter of the system on which the plans for determining the intensity of production are based. The value of  $z_3$  is obtained as the previous value of this parameter, adjusted by a certain fraction of the difference between the intensity of sales orders during the last time interval  $C[t_j - t_i]$  and the previously calculated  $z_3[t_i]$ :

$$z_3[t_j] = z_3[t_i] + \tau \left( \frac{1}{b_4} \right) (C[t_j - t_i] - z_3[t_i]) \quad (11.12)$$

Here the constant  $b_4$  is a fraction of the difference by which it is necessary to adjust the intensity of orders of the sales link. The next volumetric parameter is the value of the products at the manufacturing stage. This value is determined by the equation:

$$z_4[t_j] = z_4[t_i] + \tau (W_3[t_j - t_i] - W_2[t_j - t_i]). \quad (11.13)$$

The volume of incomplete production  $z_4[t_j]$  in accordance with the principle of a "flowing tank" is determined as the previous value of this parameter, adjusted for the difference between the planned value of the intensity of production at the input  $W_3$  and the production intensity at the output  $W_2$ .

The parameters  $z_2$  and  $z_1$  are determined similarly, which are, respectively, the value of the actual stock at the warehouse and the volume of unfulfilled production orders:

$$z_2[t_j] = z_2[t_i] + \tau (W_2[t_j - t_i] - W_1[t_j - t_i]), \quad (11.14)$$

$$z_1[t_j] = z_1[t_i] + \tau (C[t_j - t_i] - W_1[t_j - t_i]). \quad (11.15)$$

Let turn to the equations of intensities. Since in the block "Sales orders" an external demand flow is formed, determined by the sales link, we start with a description of the processes in the block "Production intensity".

The production intensity is determined by the following equation:

$$W_3[t_k - t_j] = \begin{cases} g_1, & \text{if } g_1 \leq b_6 \\ b_6, & \text{if } g_1 > b_6 \end{cases} \quad (11.16)$$

where  $g_1 = C[t_{j+1} - t_j] + \frac{1}{b_5} (b_3 z_3[t_j] - z_2[t_{j-1}])$  - planned production capacity;  $b_3$  -

coefficient of proportionality between the averaged sales order and the desired stock of production warehouse.

If the marginal production capacity  $b_6$  turns out to be less than planned, then the value of the intensity  $W_3$  is equal to the left side of the inequality that is valid in the particular case.

The meaning of this equation is quite transparent. Obviously, the intensity of production will primarily depend on the intensity of sales orders  $C[t_j-t_i]$ . The second term of the equation is a correction of the existing intensity depending on the difference between the desired  $b_5z_3[t_j]$  and actual  $z_2[t_i]$  stocks, taking into account the delay  $b_5$ , which is the reaction of the system to the occurrence of a shortage of goods on stock. From the expression (11.16) it follows that these considerations are valid only if the maximum production capacity  $b_6$  is not exceeded.

In order to determine the intensity of product output, the first-order delay equation is used:

$$W_2[t_k - t_j] = z_4[t_j]/(b_8) \quad (11.17)$$

where  $b_8$  – the delay associated with the duration of the production time cost.

In order to determine the intensity of the products supply from production to the sales link, it is necessary to take into account a number of factors related not only to the availability of actual stock, but also to the amount of accumulated unfulfilled orders, as well as delays of transport and organizational character. Based on the foregoing, to calculate the intensity of products supply from production to the sales link we obtain the expression :

$$W_1[t_k - t_j] = \begin{cases} g_2, & \text{if } g_2 \leq \frac{z_2[t_j]}{\tau} \\ \frac{z_2[t_j]}{\tau}, & \text{if } \frac{z_2[t_j]}{\tau} < g_2 \end{cases} \quad (11.18)$$

where  $g_2 = \frac{z_1[t_j]}{(b_1 + b_2b_3z_3[t_j]/z_2[t_j])}$ ;  $b_1$  – minimum delay in order fulfillment by the sales link;  $b_2$  – order fulfillment delay of organizational character.

The value of the intensity  $W_1[t_k-t_j]$  is equal to the left side of the inequality that is valid in the particular case.

Let obtain the equations of processes on the basis of the technological scheme shown on the figure 11.9. As follows from the technological scheme, there are several relatively isolated block chains corresponding to various stages of the process. Let start by obtaining equations for a blocks chain that ensure the fulfillment of orders both from stocks and from production. So the equation for determining  $d_1$ , which is in the process of registration at the plant, in accordance with the principle of "flowing tank", has the form:

$$d_1[t_j] = d_1[t_i] + \tau(f_1[t_j - t_i] - f_2[t_j - t_i] - f_3[t_j - t_i]) \quad (11.19)$$

Equation (11.19) determines the number of orders received by production, but not yet fulfilled.

Upcoming deliveries on orders at the expense of stocks that are already issued but not yet complete are determined by the formula:

$$d_2[t_j] = d_2[t_i] + \tau(f_2[t_j - t_i] - f_4[t_j - t_i]) \quad (11.20)$$

The order shipment intensity will be presented as a fixed part of the debt on orders  $d_2$ . Determining it with the help of exponential delay of the first order, we have:

$$f_4[t_j - t_i] = \frac{d_2[t_j]}{\tau_1} \quad (11.21)$$

where  $\tau_1$  – products shipment delay from production (in units of time).

Stock of finished products in production is determined by the equation

$$d_3[t_j] = d_3[t_i] + \tau(f_5[t_j - t_i] - f_4[t_j - t_i]) \quad (11.22)$$

The purpose of the stock is to ensure the delivery of products on orders within a certain time in the absence of new product receipts.

We now present an expression for determining the intensity of orders satisfied at the expense of production stocks

$$f_2[t_k - t_j] = \tau_2 \frac{d_1[t_j]}{\tau} \quad (11.23)$$

where  $\tau_2$  – a part of the total number of orders that are satisfied at the expense of production stocks (dimensionless quantity).

In order to determine  $d_4$  let create the following equation

$$d_4[t_j] = d_4[t_i] + \frac{\tau}{\tau_3} (f_4[t_j - t_i] - d_4[t_j - t_i]) \quad (11.24)$$

where  $\tau_3$  – shipment averaging time (in units of time). This time parameter plays a significant role in planning production volumes for replenishment of warehouse stocks.

The intensity of orders carried out by production is determined by that part of orders that cannot be satisfied at the expense of the stock.

$$f_3[t_k - t_j] = (1 - \tau_2) \frac{d_1[t_j]}{\tau} \quad (11.25)$$

In order to determine  $d_7$  we compose the following equation

$$d_7[t_j] = d_7[t_i] + \tau(f_3[t_j - t_i] - f_7[t_j - t_i]) \quad (11.26)$$

Then taking into account  $\tau_4$  the equation for the intensity of the sales orders shipment has the form:

$$f_7[t_k - t_j] = d_7[t_j] / \tau_6 \quad (11.27)$$

where  $\tau_4$  – production delay (in units of time).

Now we turn to the description of the blocks chain that provide for the replenishment of stock. There are three main factors that determine stock replenishment rules. The first factor is the average intensity of stock reductions due to deliveries to consumers. The second factor is the regulation of the actual stock in order to bring it into line with the desired level. The third factor is the identification of the number of orders sent for stock replenishment and being in production, taking into account the production time for these orders. Based on these factors, we define the expression for the orders intensity on stock replenishment

$$f_6[t_k - t_j] = \begin{cases} g_2, & \text{if } g_3 \geq 0 \\ 0, & \text{if } g_3 < 0 \end{cases} \quad (11.28)$$

where  $g_3 = d_4[t_j] + \frac{1}{\tau_4}(\tau_3 d_5[t_j] - d_3[t_j])$ ,  $\tau_5$  – stock control time at the plant (in units of time);

Coefficient  $\tau_6$  is the number in units of time over which the desired level of stock could ensure the fulfillment of demand for products at a given, steady-state order flow intensity.

In order to determine  $d_5$ , we create the equation

$$d_5[t_j] = d_5[t_i] + \frac{\tau}{\tau_7}(f_1[t_j - t_i] - d_5[t_j - t_i]) \quad (11.29)$$

where  $\tau_7$  – time of averaging orders in production (in units of time). This averaging of the intensity of incoming orders is necessary due to significant fluctuations in the order flow from one time interval to another, for example, equation (11.31) shows that during each time interval a certain part of the difference between the current and average sales is used to adjust the level of average sales.

The next two blocks, characterized by the parameters  $d_6$  and  $f_5$ , complete this chain. The structure of these equations is similar to equations (11.26) - (11.27) for the production of products for the implementation of direct sales orders and has the form

$$d_6[t_j] = d_6[t_i] + \tau(f_6[t_j - t_i] - f_5[t_j - t_i]) \quad (11.30)$$

$$f_5[t_k - t_j] = d_6[t_j] / \tau_6 \quad (11.31)$$

The next blocks chain, shown on the figure 11.9, is the blocks that reflect the process of ensuring the production with basic raw materials.

Obviously, that the initial data for the efficient provision of the production with the necessary amount of raw materials is  $d_5$ . And  $d_8$  is determined by the intensity of their receipt on production and the intensity of the use of materials for the production of products both for the formation of stock and at the request of consumers:

$$d_8[t_j] = d_8[t_i] + \tau(f_8[t_j - t_i] - f_3[t_j - t_i] - f_6[t_j - t_i]) \quad (11.32)$$

The value  $f_9$  depends on the intensity of raw materials consumption by the production, as well as on the conditions of regulation of stocks and contents of supply:

$$f_9[t_k - t_j] = f_3[t_j - t_i] + f_6[t_k - t_j] + \frac{1}{\tau_9}(d_5[t_j]\tau_8 - d_8[t_i] + d_5[t_j]\tau_{10} - d_9[t_i]) \quad (11.33)$$

$\tau_8$  – coefficient characterizing the stock of materials at the plant (in units of time);  $\tau_9$  – regulation time of stocks of materials at the plant (in time units);  $\tau_{10}$  – delayed receipt of materials at the plant (in units of time);

In (11.33)  $f_9$  depends on the orders intensity  $f_3$  at that moment in production and the orders intensity for stock replenishment  $f_6$ . The expression in parentheses is the sum of two differences: between the desired and actual stocks of materials and between the necessary and actual quantities of materials in the supply channels. The time constant  $\tau_{10}$  determines the intensity of the discrepancy correction between the desired and actual values. The required number of orders for materials and the number of materials on the way in the supply channels are proportional to the average level of business activity and the length of these channels.

In order to determine  $d_9$ , we use the following expressions

$$d_9[t_j] = d_9[t_i] + \tau(f_9[t_j - t_i] - f_8[t_j - t_i]), \quad (11.34)$$

$$f_8[t_k - t_j] = d_9[t_j] / \tau_{17}. \quad (11.35)$$

Intensity equations determine the intensity of flows between the volume indicators in the system and are “decision functions”. Intensity equations are solved

on the basis of data on the currently existing values of volume indicators in the block from which the flow with a given intensity proceeds, and the volume indicator to which the flow is directed. Regarding the equations of intensities, it is important to note that they regulate the actions that must occur in the system over the next time interval. At the moment of time  $t_j$ , the intensity equation is solved in order to determine the action that will control the flow intensity during the forthcoming time interval  $[t_j, t_k]$ . The group of intensity equations is solved after the equations for calculating the volume indicators are solved.

As it can be seen from the above technological schemes, an important factor in the VC processes is the delay, which can be determined using functions of various kinds. Here we will consider only one class of delay functions — exponential, which are simple in form and sufficiently fully reflect the functioning of real systems.

The exponential delay of the first order (Figure 11.8) consists of a level (which absorbs the difference between the intensities of the incoming and outgoing flows) and the intensity of the outgoing flow, which depends on the values of the level and the average delay (constant); the intensity of the incoming flow is determined by other parts of the system.

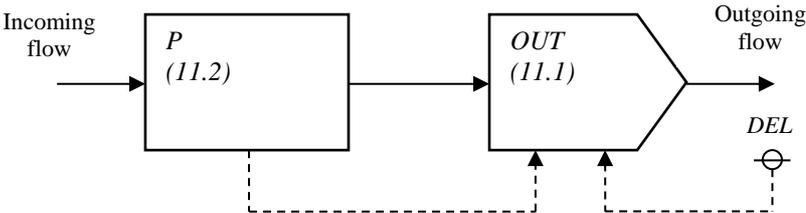


Figure 11.8 – Exponential delay of the first order

The output flow intensity in accordance with the definition of this class of delays is equal to the level divided by the average delay:

$$OUT[t_k - t_j] = \frac{P[t_j]}{DEL}, \tag{11.36}$$

where  $OUT$  – output flow intensity (units/time);  $P$  – the level in delay (units);  $DEL$  – is the delay constant, representing the average time required to overcome the delay.

Equation 11.36 is an intensity equation and determines the so-called “implicit” decision, since it is not taken on the instructions of the head, but follows from the current state of the system displayed by the variable level  $P$ .

The delay representation is incomplete as long as there is no equation for determining the quantity of  $P$  moving inside the delay. The level of  $P$  in the delay is accumulated due to the difference in the intensities of the incoming and outgoing flows:

$$P[t_j] = P[t_i] + \tau(IN[t_j - t_i] - OUT[t_j - t_i]), \quad (11.37)$$

where  $P$  – level in delay (units);  $\tau$  – the interval between successive solutions of the equation (time);  $IN$  – the intensity of the input flow specified by another equation of the system (units/time);  $OUT$  – the intensity of the outgoing flow (units/time).

High order exponential delays are obtained by conducting a flow through two or more successive first order delays. Delays of the first and higher orders can have the same total average delay  $D$ , but they will differ in unsteady reaction to changes in the flow intensity.

The figure 11.9 shows the third-order delay in the form of the total delay  $DEL$ . This total delay is distributed on three equal parts, each of which is a first order delay.

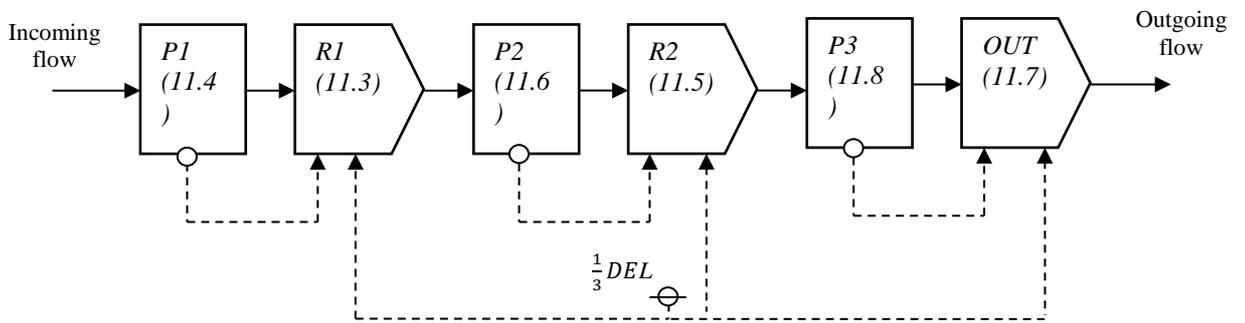


Figure 2.9 – Exponential delay of the third order

The exponential delay of the third order is determined by three pairs of equations, similar to (11.36) and (11.37), which relate the flow intensities at the input  $IN$  and output  $OUT$ :

$$R1[t_k - t_j] = \frac{P1[t_j]}{DEL/3} \quad (11.38)$$

$$P1[t_j] = P1[t_i] + \tau(IN[t_j - t_i] - R1[t_j - t_i]) \quad (11.39)$$

$$R2[t_k - t_j] = \frac{P2[t_j]}{DEL/3} \quad (11.40)$$

$$P2[t_j] = P2[t_i] + \tau(R1[t_j - t_i] - R2[t_j - t_i]) \quad (11.41)$$

$$OUT[t_k - t_j] = \frac{P3[t_j]}{DEL/3} \quad (11.42)$$

$$P3[t_j] = P3[t_i] + \tau(R2[t_j - t_i] - OUT[t_j - t_i]) \quad (11.43)$$

We are more interested in the total amount of  $P$  passing through this delay than in its individual sections. In this case, you can write:

$$P[t_j] = P1[t_j] + P2[t_j] + P3[t_j] \quad (11.44)$$

Equations of the type (11.38) - (11.43) and a diagram similar to that shown on the figure 11.9 are compiled each time a third-order delay is to be represented. Further, an abbreviation will be applied in such cases. The total amount moving in the delay described by equation (11.44); this amount can be determined directly from the equation:

$$P[t_j] = P[t_i] + \tau(IN[t_j - t_i] - OUT[t_j - t_i]) \quad (11.45)$$

which is similar to the equation (11.37).

In order to present in a concise form the equations (11.38) - (11.43), we can use the following functional notation:

$$OUT[t_j - t_i] = DELAY3(IN[t_j - t_i], DEL) \quad (11.46)$$

This designation is not an equation; it only indicates that the necessary set of equations for the delay of the third order is given. Here, “*OUT*” denotes the name of the outgoing flow; “*DELAY3*” indicates that a third-order delay should be included in the flow, and the average delay value should be “*DEL*”.

To indicate third-order delay, two equations written in the form of equations (11.45) and (11.46) are sufficient.

Even the most advanced organizational system requires constant control and corrective decisions, which is caused by constant changes in the internal and external environment in which the system operates. The greatest changes occur in the external environment, primarily under the influence of factors such as the crisis, sharp fluctuations in exchange rates and market conditions, the development of new technologies, innovations, etc.

The above mentioned factors in total lead to the need to monitor, control and constant design of supply chains even in the already established "stable" chains. In order the supply chain meets the changes, flexibly adapting to circumstances, it is necessary to use the “methodology of systematic planning and logistics system design”, which would allow taking into account emerging circumstances and evaluate alternatives for future developments.

The developed VC is characterized by two types of variables, depending on what they determine - values and intensity indicators.

### 11.3 VC functioning algorithm

Let create an algorithm to determine the main parameters of the VC functioning process [41].

Step 1. Modeling or forecasting of the demand  $D$ .

Step 2. Determining the average sale of goods

$$x_7[t_j] = x_7[t_i] + \tau \left( \frac{1}{a_4} \right) * (S[t_j - t_i] - x_7[t_i])$$

Step 3. Determining the level of unfulfilled orders

$$x_1[t_j] = x_1[t_i] + \tau (S[t_j - t_i] - V_1[t_j - t_i])$$

Step 4. Determining the actual level of stock in the sales link

$$x_2[t_j] = x_2[t_i] + \tau (V_2[t_j - t_i] - V_1[t_j - t_i])$$

Step 5. Calculating the desired stock

$$x_6[t_j] = a_3 x_7[t_j]$$

Step 6. Calculating the variable delay in the fulfillment of orders by sales link

$$x_4[t_j] = x_1[t_j] / x_4[t_j]$$

Step 7. Calculating an auxiliary variable corresponding to the expected intensity

$$x_3[t_j] = x_1[t_j] / x_4[t_j]$$

Step 8. Calculating the maximum shipping intensity

$$x_5[t_j] = x_2[t_j] / \tau$$

Step 9. Let move to to the variables describing the delays. We consider three separate delays:

a) in the placement of orders - the amount of goods in transit

$$x_{11}[t_j] = x_{11}[t_i] + \tau (V_3[t_j - t_i] - V_4[t_j - t_i])$$

b) during the transfer of orders from the sales link to production - the

amount of goods issued by the sales link to purchase located in the postal channels

$$x_{12}[t_j] = x_{12}[t_i] + \tau(V_4[t_j - t_i] - C[t_j - t_i])$$

c) during the delivery of goods from production to the sales link - goods on the way to the sales link

$$x_{13}[t_j] = x_{13}[t_i] + \tau(W_1[t_j - t_i] - V_2[t_j - t_i])$$

Step 10. Calculating the average intensity of production sales

$$z_7[t_j] = z_7[t_i] + \tau(1/b_4)(C[t_j - t_i] - z_7[t_j - t_i])$$

Step 11. Calculating the value of unfulfilled production orders

$$z_1[t_j] = z_1[t_i] + \tau(C[t_j - t_i] - W_1[t_j - t_i])$$

Step 12. Calculating the value of available stocks at production warehouses

$$z_2[t_j] = z_2[t_i] + \tau(W_2[t_j - t_i] - W_1[t_j - t_i])$$

Step 13. Calculating the value of orders in the process of registration on the production

$$z_{11}[t_j] = z_{11}[t_i] + \tau(W_4[t_j - t_i] - W_5[t_j - t_i])$$

Step 14. The production of finished goods  $z_{12}$  will depend on the orders received and will change after the change in the intensity of orders receipt with some delay

$$z_{12}[t_j] = z_{12}[t_i] + \tau(W_5[t_j - t_i] - W_2[t_j - t_i])$$

Step 15. Calculating the desired stock of production

$$z_6[t_j] = b_3 z_7[t_j]$$

Step 16. Calculating the delay in the fulfillment of orders by production

$$z_4[t_j] = b_1 + b_2(z_6[t_j]/z_2[t_j])$$

Step 17. Calculating the verified intensity of production supplies

$$z_3[t_j] = \frac{z_1[t_j]}{z_4[t_j]}$$

Step 18. Calculating the maximum intensity of production supplies

$$z_5[t_j] = z_2[t_j] / \tau$$

Step 19. Determining the desired level of orders and goods on the channels connecting the sales link and production

$$x_8[t_j] = (x_7[t_j])(a_6 + a_7 + z_4[t_j] + a_8)$$

Step 20. Determining the actual level of orders issued by the sales link on the channels consists of the sum of orders and goods in various sections of the channels

$$x_9[t_j] = x_{11}[t_j] + x_{12}[t_j] + z_1[t_j] + x_{13}[t_j]$$

Step 21. Finding a normal level of unfulfilled sales orders

$$x_{10}[t_j] = x_7[t_j](a_1 + a_2)$$

Step 22. Calculating the desired level of stocks transferred through production channels

$$z_8[t_j] = z_7[t_j](b_7 + b_8)$$

Step 23. Calculating the actual level of stocks transferred through production channels

$$z_9[t_j] = z_{11}[t_j] + z_{12}[t_j]$$

Step 24. Calculating normal volume for production of unfulfilled orders

$$z_{10}[t_j] = z_7[t_j](b_1 + b_2)$$

Step 25. Determining the intensity of supplies from production warehouses

$$W_1 = \begin{cases} z_3[t_j], & \text{if } z_5[t_j] \geq z_3[t_j] \\ z_5[t_j], & \text{if } z_5[t_j] < z_3[t_j] \end{cases}$$

Step 26. Calculating the desired production intensity

$$W_3 [t_k - t_j] = C[t_j - t_i] + \frac{1}{b_5} (z_6[t_j] - z_2[t_j] + z_8[t_j] - z_9[t_j] + z_1[t_j] - z_{10}[t_j])$$

Step 27. Calculating the intensity of goods production as the smallest of the two intensities - the desired and limited ones by the production capacity

$$W_4[t_k - t_j] = \begin{cases} W_3[t_k - t_j], & \text{if } b_6 \geq W_3[t_k - t_j] \\ b_6, & \text{if } b_6 < W_3[t_k - t_j] \end{cases}$$

Step 28. Determining the intensity of goods shipment from the sales link

$$V_1 = \begin{cases} x_3[t_j], & \text{if } x_5[t_j] \geq x_3[t_j] \\ x_5[t_j], & \text{if } x_5[t_j] < x_3[t_j] \end{cases}$$

Step 29. Determining the intensity of outgoing orders from the sales link

$$V_3[t_k - t_j] = S[t_j - t_i] + \frac{1}{a_5} (x_6[t_j] - x_2[t_j] + x_8[t_j] - x_9[t_j] + x_1[t_j] - x_{10}[t_j])$$

Step 30. The intensity of the orders issued by the sales link for the purchase of goods:

$$V_4[t_k - t_j] = DELAY3(V_3[t_j - t_i], a_5)$$

here and hereinafter *DELAY3* – functional designation (not a variable) defining a system of equations describing the delay of the third order. The exponential delay of the third order is determined by the three pairs of equations described above.

Step 31. Supply intensity in the sales chain

$$V_2[t_k - t_j] = DELAY3(W_1[t_j - t_i], a_8)$$

Step 32. Defining orders received by production

$$C[t_k - t_j] = DELAY3(V_4[t_j - t_i], a_7)$$

Step 33. Calculating the intensity of production orders at the enterprise

$$W_5[t_k - t_j] = DELAY3(W_4[t_j - t_i], b_7)$$

Step 34. Calculating the intensity of replenishment stock at the enterprise

$$W_2[t_k - t_j] = DELAY3(W_5[t_j - t_i], b_8)$$

Step 35. Analysis of the results and determination of their impact on the behavior of the system.

The modeling results of the VC functioning process according to the algorithm for the test initial data are presented on the figure 11.10. The results obtained by value chain modeling described by the model will allow verifying how the above VC will respond to certain input data.

t	S	C	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	x12	x13	v1	v2	v3	v4	z1	z2	z3	z4
0	10	10	14	80	10	1.4	1600	80	10	0	0	0	30	5	10	10	10	10	7.5	20	40	10	2
0.05	6	9.93	13.8	80	9.86	1.4	1600	79.8	9.97	64.84	65	13.96	30.12	4.87	10	9.86	10	15.95	7.5	20	40	10	2
0.1	8	9.64	13.71	80.01	9.8	1.4	1600.14	79.7	9.96	64.76	65.3	13.95	30.55	4.75	10	9.8	10	17.37	7.53	20	40	10	2
0.15	4	9.42	13.42	80.02	9.6	1.4	1600.33	79.4	9.93	64.51	65.67	13.9	31.04	4.65	10	9.6	10	12.18	7.64	19.98	40	9.99	2
0.2	4	9.22	13.14	80.04	9.41	1.4	1600.73	79.11	9.89	64.27	65.77	13.84	31.27	4.56	10	9.41	10	11.24	7.76	19.95	40	9.98	2
0.25	6	9.05	12.97	80.07	9.3	1.39	1601.32	78.91	9.86	64.1	65.84	13.81	31.44	4.49	10	9.3	10	12.63	7.82	19.91	40	9.96	2
0.3	6	8.92	12.8	80.1	9.19	1.39	1602.02	78.72	9.84	63.94	65.97	13.78	31.68	4.42	10	9.19	10	11.95	7.86	19.87	40	9.94	2
0.35	10	8.8	12.84	80.14	9.22	1.39	1602.82	78.73	9.84	63.94	66.07	13.78	31.88	4.37	9.99	9.22	10	15.86	7.92	19.82	40.01	9.92	2
0.4	4	8.71	12.58	80.18	9.04	1.39	1603.6	78.44	9.8	63.69	66.36	13.73	32.28	4.33	9.99	9.04	9.99	8.74	7.97	19.76	40.01	9.9	2
0.45	10	8.63	12.63	80.23	9.08	1.39	1604.55	78.44	9.81	63.69	66.3	13.73	32.32	4.29	9.99	9.08	9.99	14.81	8.07	19.7	40.02	9.87	2
0.5	6	8.56	12.47	80.27	8.97	1.39	1605.46	78.25	9.78	63.53	66.54	13.69	32.66	4.26	9.98	8.97	9.98	10.02	8.08	19.64	40.03	9.85	1
0.55	8	8.51	12.43	80.32	8.94	1.39	1606.47	78.17	9.77	63.44	66.54	13.68	32.75	4.24	9.97	8.94	9.98	11.75	8.17	19.57	40.05	9.82	1
0.6	8	8.46	12.38	80.38	8.91	1.39	1607.51	78.08	9.76	63.36	66.63	13.66	32.93	4.22	9.96	8.91	9.97	11.4	8.19	19.51	40.06	9.79	1
0.65	8	8.43	12.33	80.43	8.89	1.39	1608.56	77.99	9.75	63.27	66.7	13.65	33.09	4.21	9.96	8.89	9.96	11.06	8.24	19.44	40.08	9.77	1
0.7	4	8.41	12.09	80.48	8.72	1.39	1609.64	77.7	9.71	63.03	66.75	13.6	33.23	4.2	9.95	8.72	9.95	6.18	8.27	19.37	40.1	9.74	1
0.75	6	8.39	11.95	80.54	8.63	1.38	1610.87	77.52	9.69	62.86	66.57	13.57	33.13	4.19	9.94	8.63	9.94	7.83	8.31	19.31	40.13	9.71	1
0.8	8	8.38	11.92	80.61	8.61	1.38	1612.19	77.43	9.68	62.78	66.46	13.55	33.11	4.19	9.92	8.61	9.93	9.67	8.28	19.24	40.16	9.69	1
0.85	6	8.37	11.79	80.68	8.53	1.38	1613.51	77.25	9.66	62.61	66.45	13.52	33.18	4.18	9.91	8.53	9.92	7.15	8.28	19.18	40.19	9.66	1
0.9	8	8.36	11.76	80.75	8.51	1.38	1614.9	77.16	9.65	62.53	66.31	13.5	33.12	4.18	9.9	8.51	9.91	9.03	8.29	19.11	40.23	9.64	1
0.95	4	8.35	11.54	80.82	8.36	1.38	1616.3	76.88	9.61	62.28	66.26	13.45	33.16	4.18	9.89	8.36	9.9	4.26	8.28	19.05	40.26	9.62	1
1	6	8.35	11.42	80.89	8.28	1.38	1617.84	76.7	9.59	62.11	65.98	13.42	32.95	4.17	9.87	8.28	9.88	6.01	8.29	18.98	40.3	9.6	1

Figure 11.10 – Output data

### 11.4 Modeling the functioning process of a small (medium) enterprise

For the effective functioning of various production facilities in the country, including small enterprises (SE), the choice of the optimal strategy for their development in a market economy is essential. One of the most important factors in the functioning and development of these facilities is the effective use of credit or borrowed funds. To solve these problems, the use of classical economic and mathematical models does not lead to the desired results due to the high degree of non-stationarity and stochasticity of the main parameters and indicators of this industry. In the section on the basis of well-known approaches to the creation of simulation models of production, there is proposed a generalized simulation model of the functioning of a small enterprise to analyze the influence of non-stationarity and randomness of demand and supply on the efficiency of its operation [59].

### 11.4.1 Functional structure of production and sales of SE products

The functional structure of a small enterprise as a system is shown on the figure 11.11 and clearly reflects both production and information management functions.

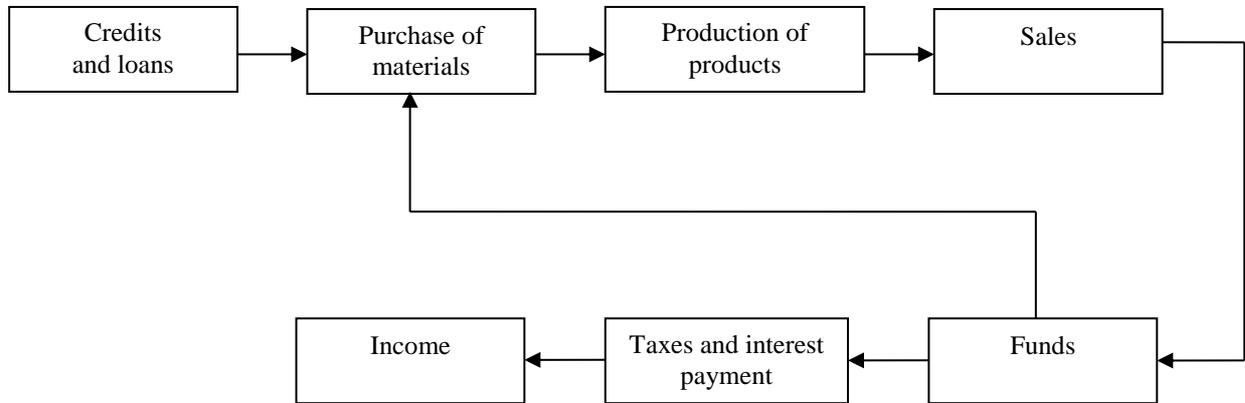


Figure 11.11 – Functional structure of the system

The intensity of production volume and products sales growth, improving its quality directly affect the final results of the functioning of the enterprise, so the analysis of all indicators of the functional structure is important.

### 11.4.2 System operation model

To describe and analyze the dynamics of the system operation, it is advisable to create an econometric model in the form of recurrence relations [42].

Let start from the creation of the equation for the “Purchase of materials” block:

$$M(t) = S(t-1) + \frac{1}{a_1} \{ZNS(t) - FZ(t)\}, \quad (11.47)$$

where  $M$  – volumes of purchased raw materials,  $ZNS$  – desired stock of raw materials,  $FZ$  – actual stock of raw materials,  $S$  – the volume of product sales,  $a_1$  – raw material stock regulation delay.

Block "Production" is described by the equation:

$$W(t) = S(t) + \frac{1}{a_2} \{JZP(t) - FZP(t)\}, \quad (11.48)$$

where  $W$  – volume of production,  $JZP$  and  $FZP$  – respectively desired and actual stocks of products,  $a_2$  – product stock regulation delay,  $t$  – current moment of time,  $t-1$  – previous moment of time.

In the block "Sales" there is carried out modeling the products sales intensity as a continuous or integer random variable.

To assess the funds we use the expression:

$$CN(t) = CN(t-1) + \frac{1}{a_2} \{S(t)TC - M(t)TCS\}, \quad (11.49)$$

where  $CN$  – cash,  $TC$  – product price,  $TCS$  – raw material price.  
The total costs of taxes and interest are determined by the formula:

$$NP(t) = b_1 CN(t) + r_L L, \quad (11.50)$$

where  $b_1$  – tax rate, and  $r_L L$  – loan interest payment  $L$ .

In order to calculate the income  $PR$  there is used the expression:

$$PR(t) = CN(t) - NP(t) \quad (11.51)$$

All the presented equations make up the model of operation of the studied system.

#### 11.4.3 The dynamic model of the analysis of SE activity.

The determining factor in the life of a small enterprise, obviously, is the efficiency of the use of funds. We describe the equations characterizing changes in money and loans:

$$\frac{d(CN)}{dt} = S_f(t) - M(t) + H(t);$$

$$\frac{dL}{dt} = r_L L(t) + H(t).$$

where  $S_f(t)$  – the volume of sales in monetary terms  $S_f(t) = S(t)TC$ ,  $r_L$  – bank rate,  $H(t)$  – borrowed funds.

It is assumed that to ensure the continuity of production costs, it is required to have a stock of finance [6]:

$$M(t)TCS + NP(t) \leq CN(t) / \Theta$$

where  $\Theta$  – liquidity ratio.

The profitability of a small enterprise depends on the dynamics of prices for raw materials and its products. Denote the predicted value of price inflation by  $i$ . The total discount coefficient that takes into account price inflation is denoted by

$$\Delta = \delta + \alpha i, \quad (11.52)$$

where  $\delta$  – discount coefficient, and  $\alpha \in [0,1]$ .

The income of a small enterprise, taking into account inflation of product prices, is predicted according to the exponential law with intensity  $\gamma$ .

$$S_f(t) = S_f e^{\gamma t}$$

Formulate the problem of maximizing the total income  $SS$  of a small enterprise

$$SS = \int_0^T S_f(t)^\alpha e^{-\Delta t} dt \rightarrow \max ; \quad (11.53)$$

$$\frac{dL}{dt} = rL + H ;$$

$$\frac{d(CN)}{dt} = S_f e^{\gamma t} - M_f + H ; \quad (11.54)$$

$$CN \geq M_f + NP, \quad L(t) = 0, \quad L \geq 0, \quad CN \geq 0$$

where  $M_f$  – monetary expressions of materials.

On the basis of the optimal trajectory of the problem (11.18, 11.19)

$$CN^*(t) = \Theta[M_f^*(t) + NP^*(t)], \quad \forall t \in [0, T], \quad (11.55)$$

obtained by analogy with the optimization of consumer expenditures, considered in [7], we can give the following economic interpretations of the results of the small enterprise activity. Depending on the discount coefficient, small enterprises may have three variants of the result of their activities.

1) If the relation is satisfied:

$$\Delta > r + (1 - \alpha) / \Theta$$

a small enterprise is constantly forced to take loans. With this level of discount coefficient, a small enterprise is an unprofitable economic entity.

2) If the relation is satisfied:

$$r - (1 - \alpha)\gamma < \Delta < r + (1 - \alpha) / \Theta$$

a small enterprise can both attract loans and operate at its own funds in the short term. But over time, in the absence of funding, a small enterprise also becomes unprofitable.

3) If the relation is satisfied:

$$\Delta < r - (1 - \alpha)\gamma$$

a small enterprise can both take loans and spend only its own funds. Under any initial conditions, it pays off loan obligations for a limited period of time. Loans are needed only to quickly reach the optimal mode of operation of a small enterprise.

The developed model is focused mainly on the application in the tasks of prospective analysis, when it is necessary to find out the admissible possible variants for the development of the studied economic objects. Also, the proposed model, in condition of its appropriate modification, may be suitable not only for forecasting, but also for the current management of the small enterprise.

#### 11.4.4 The algorithm of a small (medium) enterprise functioning

The figure 11.12 shows a scheme of a simulation modeling of the process of a small (medium) enterprise functioning.

The algorithm for modeling the process of a small enterprise functioning includes the following steps:

Step 1. Modeling the volumes and vector of time points for receipt of credits and loans.

Step 2. Calculating volumes of purchased materials.

$$M(t) = S(t-1) + \frac{1}{a_1} \{ZNZ(t) - FZ(t)\}$$

Step 3. Calculating production volumes.

$$W(t) = S(t) + \frac{1}{a_2} \{JZP(t) - FZP(t)\}$$

Step 4. Modeling sales volumes as continuous or integer random variables.

Step 5. Calculating cash value:

$$CN(t) = CN(t-1) + \frac{1}{a_2} \{S(t)TC - M(t)TCS\}$$

Step 6. Calculating total tax costs and interest payments.

$$NP(t) = b_1 CN(t) + r_L L$$

Step 7. Calculating income.

$$PR(t) = CN(t) - NP(t)$$

Step 1. Analysis of the compliance of steps 1-5 by the formulas the values of the parameters of the SE activity with the optimal trajectory

Step 2.

$$CN^*(t) = \Theta[M_f^*(t) + NP^*(t)], \quad \forall t \in [0, T], \quad (11.56)$$

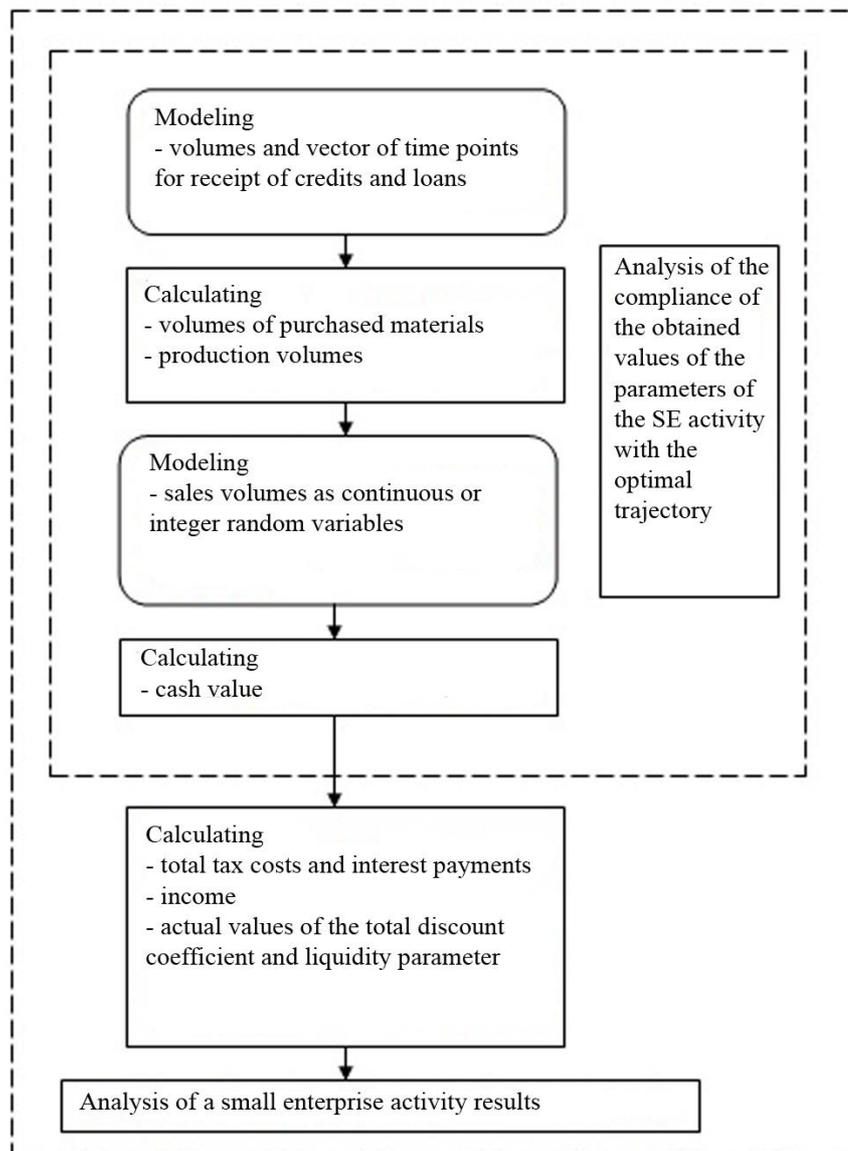


Figure 11.12 – A scheme of a small (medium) enterprise functioning

Step 3. Analysis of the compliance of steps 1-5 by the formulas the values of the parameters of the SE activity with the optimal trajectory ,

Step 4.

$$CN^*(t) = \Theta[M_f^*(t) + NP^*(t)], \quad \forall t \in [0, T], \quad (11.56)$$

Tasks:

$$SS = \int_0^T S_f(t)^\alpha e^{-\Delta t} dt \rightarrow \max ;$$

$$\frac{dL}{dt} = rL + H ;$$

$$\frac{d(CN)}{dt} = S_f e^{\alpha} - M_f + H;$$

$$CN \geq M_f + NP, L(t) = 0, L \geq 0, CN \geq 0.$$

Step 5. Determination from expression (11.56) of the actual values of the total discount coefficient and liquidity parameter.

Step 6. Analysis of the performance of a small enterprise under the conditions described in chapter 11.2.3.

Step 7. Analysis of the results and the development of specific steps to adjust the strategy and to make managerial decisions.

The proposed algorithm for evaluating the effectiveness of a small enterprise allows to give an objective, structured economic evaluation of its effectiveness. The developed algorithm will allow the heads to take an economically balanced approach to the process of organizing entrepreneurial activities, to quickly make managerial decisions, and also to adjust the direction of the small enterprise as a whole based on an analysis of the effectiveness of its functioning, taking into account economic environment factors.

## **11.5 Industry model (for example, the production of petroleum products)**

Significant for the effective functioning of various sectors of the country, including oil refining, is the choice of the optimal strategy for its development in a market economy. One of the most important factors for the functioning and development of this industry is to ensure the optimal balance between demand and the production of petroleum products. In this section, on the basis of well-known approaches to creation of simulation models of production, there is proposed a generalized simulation model of the oil refining industry to analyze the effects of non-stationarity and randomness of demand and proposals on the efficiency of the industry [43].

### **11.5.1 Functional structure of the system for the production and sale of petroleum products**

The functional structure of the system is shown on the figure 11.13 and clearly reflects both production and information managerial functions.

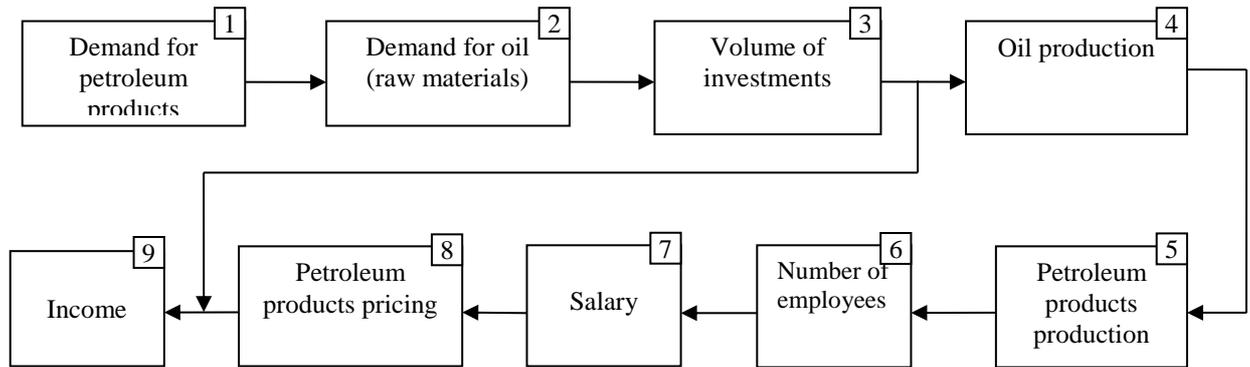


Figure 11.13 – Functional structure of the system

Demand modeling for petroleum products can be implemented in two ways: analytical and statistical. The first way involves obtaining econometric dependencies of demand on various factors, where demand in the previous period and retail price indices play a significant role. Blocks 2-5 are related to the level of fixed assets of the industry, which indirectly determine the functions of the remaining four blocks (6-9). It should also be noted that there is a significant correlation between the functions of all links of the given structure.

#### 11.5.2 Generalized indicators

Before developing a system model, we consider generalized indicators of the industry functioning. As follows from the functional structure, the system under consideration consists of two divisions: oil production and petroleum products production. The first division supplies raw materials for the second one. We give analytical dependencies to describe the generalized indicators of these divisions. Let  $\Phi_1(t)$  – the main production assets in cost terms of the first division, and  $\Phi_2(t)$  – the general production assets of the second division.  $x_1$  – the cost of the final product (oil) of the first division, and  $x_2$  – the cost of the final product (petroleum) of the second division.  $N_1$  and  $N_2$  – the flows of investment to these divisions. We write expressions for the production functions of these divisions:

$$x_1 = k_1[\Phi_1(0) + \int_0^t N_1 dt];$$

$$x_2 = k_2[\Phi_2(0) + \int_0^t N_2 dt].$$

where  $k_1$  and  $k_2$  – return on assets coefficients.

The control parameter is  $\lambda$  – a share from the total amount of savings allocated to the first division. We assume that

$$N_1 = \lambda x_1, \quad N_2 = (1 - \lambda)x_2, \quad 0 \leq \lambda \leq 1.$$

The following cybernetic scheme can be put in accordance with the obtained dependencies (Figure 11.14) [59].

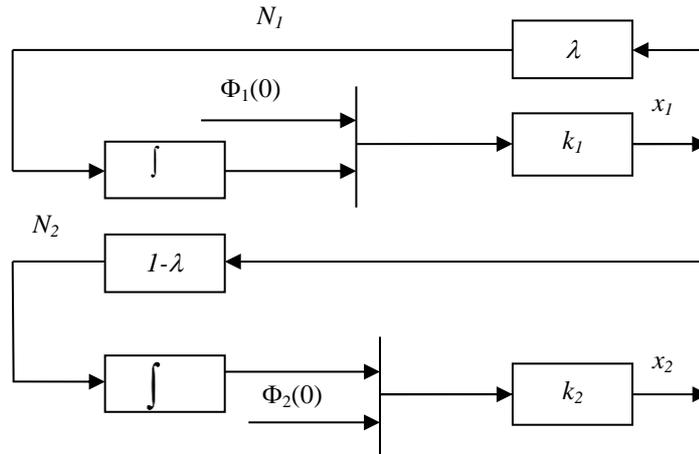


Figure 11.14 – Cybernetic scheme of the system

Using expressions of transfer functions for sequential connection and feedback, we obtain the operator form for  $x_1$  and  $x_2$ .

$$x_1(p) = \frac{k_1 \Phi_1(0)}{1 - \frac{k_1 \lambda}{p}}$$

$$x_2(p) = k_2 \left[ \Phi_2(0) + \frac{(1 - \lambda)x_1(p)}{p} \right].$$

Transforming these expressions we get:

$$x_2(p) = k_2 \Phi_2(0) + \frac{k_1 k_2 (1 - \lambda) \Phi_1(0)}{p - k_1 \lambda} = K_{11}(p) \Phi_2(0) + K_{12}(p) \Phi_1(0),$$

$$\text{where } K_{11}(p) = k_2, \text{ and } K_{12}(p) = \frac{k_1 k_2 (1 - \lambda)}{p - k_1 \lambda}.$$

Then the transition from the operator form to the original one leads to the expression&

$$x_2(t) = k_2 \Phi_2(0) + \frac{k_2 (1 - \lambda)}{\lambda} [e^{k_1 \lambda t} - 1] \Phi_1(0).$$

From this expression it is clear that the cost of production assets at the initial moment ( $t = 0$ ) leads to an increase in the production of petroleum products for all  $t > 0$ . The important role of the parameter  $\lambda$  is also obvious, i.e. the greater part of the savings goes to oil production, the higher the intensity of development of the industry. However, the factor  $(1 - \lambda)$  for the exponential function does not allow  $\lambda$  to take too large values, since at  $\lambda = 1$  the second term of the equation may turn out to be zero. Therefore, it is important to choose the value of the parameter  $\lambda$ , which can

be analytically optimized according to various decision-making criteria, or in order to find the effective value  $\lambda$  you can use simulation methods. The generalized indicators obtained in this section should serve as a guide for analyzing the results of simulation modeling using other, for example, econometric models.

### 11.5.3 Initial conditions

Suppose that at the beginning of a certain period the industry has the general production assets in value terms  $\Phi(t)$ . Based on technical and technological indicators, the maximum return on assets ratio  $k_m$  is determined. Then for the potential production capacity of the industry we have the expression:

$$M(t) = k_m \Phi(t), \quad (11.57)$$

where  $\Phi(t) = \Phi_1(t) + \Phi_2(t)$ ,  $\Phi_1(t) = \lambda \Phi(t)$ ,  $\Phi_2(t) = (1-\lambda) \Phi(t)$ .

And for the achievable or actual production capacity of the industry, we have, respectively, the ratio

$$G(t) = k_\phi \Phi(t), \quad (11.58)$$

where  $k_\phi$  – actual return on assets coefficient.

These indicators are basic for the production characteristics of the industry at the beginning of the period under review. The industry management, on the basis of market research on demand, forms the plan for production. Let  $R(t)$  – the expression of the demand volume that determines the planned task for the production output. We carry out a preliminary assessment of the tension of the planned task.

Variant 1. If

$$R(t) > M(t),$$

then the planned task can only be fulfilled if new capacities are introduced.

Variant 2. If

$$G(t) < R(t) < M(t),$$

then the plan can be fulfilled only with the implementation of innovations that increase the return on assets coefficient.

Variant 3. If

$$R(t) \leq G(t).$$

then it is necessary to carry out activities aimed at increasing demand by improving the quality and range of products.

#### 11.5.4 System functioning model

Obviously, the starting value for determining the demand for petroleum products in the current time  $R(t)$  is the demand in the previous period  $R(t-1)$ . The difference between them is due to a number of factors, the most significant of which is the relationship between the monthly income of the population  $MD$  and the index of retail prices for consumer goods  $IS$ , as well as the cost of advertising  $A$ . The effect on demand of other unaccounted factors can be taken into account using the random variable  $W_1$  with the given distribution law. Then using the apparatus of multiple linear regression we construct the relation:

$$R(t) = R(t-1) + r_1 \left[ \frac{MD(t-1)}{IS(t-1)} \right] + r_2 [A(t-1)] + W_1, \quad (11.59)$$

The block "Petroleum products production" is determined by the recurrence ratio:

$$G(t) = G(t-1) + g_1 [AR(t)] + g_2 [ZN(t-1)], \quad (11.60)$$

where  $G$  – petroleum product index,  $AR$  – average volume of sales of petroleum products for the previous time interval (for example, three months),  $ZN$  – stocks of petroleum products at points of sale in the previous period.

The block "Demand for oil (raw materials)" is described by the equation:

$$H(t) = H(t-1) + h_1 [AG(t)] + h_2 [IP(t-1)] + W_2. \quad (11.61)$$

Here  $H$  – demand value,  $AG$  – average petroleum product production index for the previous six months,  $IP$  – durable goods production index,  $W_2$  – random variable with a given distribution law, taking into account the influence on the value of demand of other unaccounted factors.

The equation for determining investment  $K$  has the form

$$K(t) = K(t-1) + k_1 [AH(t)] + k_2 T, \quad (11.62)$$

where  $AH$  – average monthly oil sales for the previous time interval (for example, six months),  $T$  – trend.

The oil production index  $Q$  depends on the demand for it, the ratio of stocks in oil storage tanks to the volume of current orders for its supply  $INV/UO$  in the previous period, season  $M$ , trend  $T$  and is determined by the expression:

$$Q(t) = Q(t-1) + q_1 [H(t)] + q_2 \left[ \frac{INV}{UO}(t-1) \right] + q_3 M + q_4 T, \quad (11.63)$$

The number of employees  $N$  employed in the industry is determined by its production in this and in the previous three months  $[AQ(t-i)]$ , as well as by the ratio between the monthly income of the population  $MD$  and the index of retail prices for consumer goods  $IS$ , which characterizes real incomes of the population in the previous period:

$$N(t) = N(t-1) + n_1[Q(t)] + n_2[AQ(t-i)] + n_3\left[\frac{MD}{IS}(t-1)\right], \quad (11.64)$$

In order to determine the salary  $E$ , in addition to the number of employees, it is necessary to take into account the value of the trend:

$$E(t) = E(t-1) + e_1[N(t)] + e_2T \quad (11.65)$$

The current price index  $P$  for petroleum products is determined by the ratio of its stocks to  $INV/VO$  orders in the previous period, by the average salary  $E$  for the industry, by previous values of the price indices for raw materials  $B$  and the trend  $T$

$$P(t) = P(t-1) + p_1\left[\frac{INV}{VO}(t-1)\right] + p_2[E(t)] + p_3[B(t-1)] + p_4T, \quad (11.66)$$

Industry model equations completes the expression for calculating income  $D$

$$D(t) = D(t-1) + d_1[P(t)] + d_2[P(t-1)] + d_3[H(t)] + d_4\left[\frac{INV}{VO}(t-1)\right] + d_5[E(t)] \quad (11.67)$$

Here, in addition to the petroleum product price index ( $P$ ), the price and volume of oil sales are also taken into account. The directly considered cost factor is the salary and costs associated with the raw materials, which are indirectly reflected in the value of stocks and appear in relation to stocks for orders.

All coefficients ( $r, g, h, k, q, n, e, p, d$ ) in models (11.59-11.67) are determined by standard regression analysis apparatus.

#### 11.5.5 The algorithm of the industry model functioning

The figure 11.15 shows a simulation modeling scheme of the industry model functioning (for example, petroleum products).

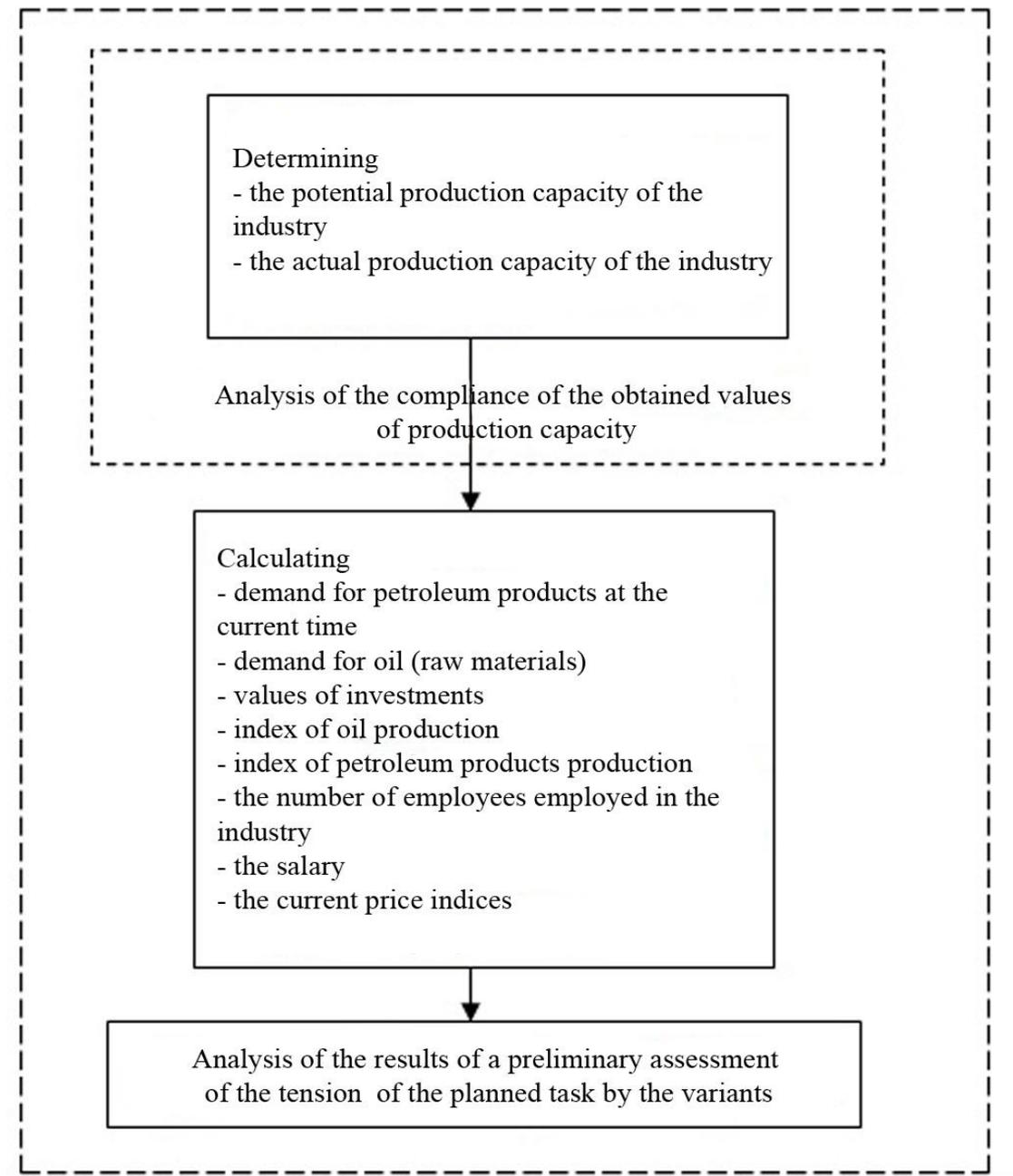


Figure 11.15 – Scheme of the functioning of production and sale of petroleum products

The algorithm for modeling the process of functioning of the production and sale of petroleum products includes the following steps [9]:

Step 1. Determining for a given value  $\lambda$  the potential production capacity of the industry:

$$M(t) = k_m \Phi(t).$$

Step 2. Determining the actual production capacity of the industry

$$G(t) = k_\phi \Phi(t). \quad (11.68)$$

Step 3. Determining the current demand for petroleum products:

$$R(t) = R(t-1) + r_1 \left[ \frac{MD(t-1)}{IS(t-1)} \right] + r_2 [A(t-1)] + W_1.$$

Step 4. Calculating the demand for oil (raw materials):

$$H(t) = H(t-1) + h_1 [AG(t)] + h_2 [IP(t-1)] + W_2.$$

Step 5. Calculating the values of investments:

$$K(t) = K(t-1) + k_1 [AH(t)] + k_2 T.$$

Step 6. Calculating the index of oil production:

$$Q(t) = Q(t-1) + q_1 [H(t)] + q_2 \left[ \frac{INV}{UO}(t-1) \right] + q_3 M + q_4 T.$$

Step 7. Calculating the index of petroleum products production:

$$G(t) = G(t-1) + g_1 [AR(t)] + g_2 [ZN(t-1)]. \quad (11.69)$$

Step 8. Calculating the number of employees employed in the industry:

$$N(t) = N(t-1) + n_1 [Q(t)] + n_2 [AQ(t-i)] + n_3 \left[ \frac{MD}{IS}(t-1) \right].$$

Step 9. Calculating the salary:

$$E(t) = E(t-1) + e_1 [N(t)] + e_2 T.$$

Step 10. Calculating the current price index:

$$P(t) = P(t-1) + p_1 \left[ \frac{INV}{UO}(t-1) \right] + p_2 [E(t)] + p_3 [B(t-1)] + p_4 T.$$

Step 11. Calculating income:

$$D(t) = D(t-1) + d_1 [P(t)] + d_2 [P(t-1)] + d_3 [H(t)] + d_4 \left[ \frac{INV}{UO}(t-1) \right] + d_5 [E(t)].$$

Step 12. Analysis of the compliance of the production capacity values obtained by formulas (11.68) and (11.69).

Step 13. Analysis of the results of a preliminary assessment of the tension of the planned tasks by the variants described in chapter 11.3.3.

Step 14. Determination of the effective value of the parameter  $\lambda$ , characterizing the distribution of the total amount of investments between oil production and petroleum products production.

Step 15. Analysis of the results and the development of specific steps to adjust the strategy and to make managerial decisions.

A study of the planning and production activities of enterprises in various industries over a considerable period of time makes it possible to create a dynamic picture of functioning in which factors such as the growth rate of the value of fixed assets, the rate of improvement in the use of fixed and circulating assets and the rate of growth of labor productivity play a significant role.

### 11.5.6 Output data

The modeling results are presented on the figure 11.16. The analysis of a preliminary assessment of the tension of the planned task showed that the planned task can be fulfilled only if new capacities are introduced.

t	Mt	Gt	Rt	Ht	Kt	Qt	GGt	Nt	Et	Pt	Dt
0.5	1350	1150	7510.32	2.01	7159.45	2051.5	9583.69	15403.58	8871.67	10494.41	17733.63
1	1350	1150	12740.28	4.68	9450.26	4322.92	25589.66	29339.09	16350.59	26877.7	40246.38
1.5	1350	1150	21421.85	7.85	25063.15	4489.31	38255.62	46051.89	25053.62	42888.1	102967.19
2	1350	1150	31203.25	9.88	33209.02	5592.65	53638.08	62596.48	70528.24	69914.17	187339.67
2.5	1350	1150	43224.55	14.02	44947.74	5614.55	60030.25	82318.33	145464.97	126154.12	364346.65
3	1350	1150	51600.53	16.84	66014.92	6402.59	66176.9	97770.4	207143.24	314322.6	797497.98
3.5	1350	1150	60185.68	19.54	74134.79	7560.68	75658.72	122139.19	293042.89	360172.8	1408476.41
4	1350	1150	68257.69	20.72	87074.77	9540.06	90097.58	143966.7	393978.7	496333.71	2205993.96
4.5	1350	1150	71444.42	24.25	95036.66	10398.52	103014.76	160068.99	526803.79	684464.72	3305270.95
5	1350	1150	73433.53	27.76	96082.53	10502.9	114575.66	179817.75	557732.68	1053136.11	4383038.67
5.5	1350	1150	84085.53	31.79	109076.7	11532.15	124646.72	192014.9	653961.88	1243380.02	5792701.09
6	1350	1150	95126.58	36.16	122714.48	11886.11	133590.44	211146.01	657538	1520874.74	8315886.72
6.5	1350	1150	108303.89	38.32	131575	13636.98	142802.44	215430.23	750046.14	2249032.72	10349385.75
7	1350	1150	123126.41	39.25	136922.15	14002.82	151655.89	236481.62	901303.17	2933084.94	15489999.23
7.5	1350	1150	130712.68	42.55	152645.23	14136.4	163786.56	251796.87	1061549.12	3677142.46	19527468.09
8	1350	1150	145221.2	45.39	157113.02	15353.03	176996.63	272518.15	1167726.59	4796230.34	21954474.7
8.5	1350	1150	148638.01	47.71	177685.42	16685.08	191592.29	290303.43	1325956.87	5986067.76	32796545.5

Figure 11.16 – Output data

## 11.6 Market analysis of goods and products

### 11.6.1 Demand modeling

Demand modeling for products depending on situations and initial data can be carried out in two ways [57, 58].

In the first case, patterns of demand change are assumed to be known, for example, depending on the season (for agricultural products or petroleum products) or depending on other factors. Then demand modeling is carried out by generating values according to the given distribution laws. In the second case, demand is significantly affected by changes of product prices. A large influence on the level and dynamics of prices has the state of the financial and credit sphere, while the direct influence on prices is exerted by changes in the purchasing capacity of the country's monetary unit.

Let start with the first case. Let demand form a random flow  $\{t_j, D_j\}$ . We will consider modeling the moments of orders  $\{t_j\}$  for a fairly general case when the set  $T = \{t_j\}$  forms a stationary Palm flow with a given density function  $\varphi(\tau)$  of intervals between elements, starting from the second interval of the flow  $T$ . In order to determine the moments  $t_j$  we use the standard formula:

$$t_j = t_{j-1} + \tau_i, \quad i = 1, 2, \dots, n$$

where  $\tau_j$  – intervals between the elements of the flow  $T$ .

If we know  $\varphi(\tau)$  it is not enough for modeling Palm flows, since the density function of the first interval is usually different from  $\varphi(\tau)$ , i.e.

$$\varphi_1(\tau) \neq \varphi(\tau)$$

Therefore, in order to determine  $\varphi_1(\tau)$  we must use the Palm formula [10]:

$$\varphi_1(\tau_1) = \lambda \left( 1 - \int_0^{\tau_1} \varphi(\tau) d\tau \right), \quad (11.70)$$

where  $\lambda$  – flow intensity.

The values of the intervals  $\tau_j$  between the elements of the flow  $T$  are determined using the method of the inverse function of continuous random variables modeling, the principle of which is formulated as a theorem [44]:

“A random variable  $\tau$  whose implementations are determined from the expression:

$$\mathcal{G}(\tau) = \int \varphi(\tau) d\tau = u \text{ or } \tau = \mathcal{G}^{-1}(u), \quad (11.71)$$

where  $u$  – a random variable uniformly distributed on the interval  $[0, 1]$  has a distribution density  $\varphi(\tau)$ ».

Or you can use the main method for discrete values modeling, based on the theorem [44]: “The quantity  $\tau_k$  given in the form of a table :

$$\begin{pmatrix} \tau_1 & \tau_2 & \dots & \tau_m \\ p_1 & p_2 & \dots & p_m \end{pmatrix}$$

occurs with probability  $p_k$  under the condition  $u \in \Delta_k$ , where  $\Delta_k = p_k \gg$ .

If the density function  $\varphi(\tau)$  of the intervals between the elements of the flow  $T$  is a continuous random variable and obeys one of the known standard theoretical distribution laws, then for modeling the values of the intervals  $\tau_j$  between the elements of the flow  $T$  you can use the formulas given in Table 11.1 [55].

The most complete list of standard theoretical distributions of continuous random variables and formulas for their modeling is given in [24].

Based on the foregoing, there was created an algorithm for consumer demand modeling, including the following steps.

Step 1. Input data set.

Step 2. Determination by the Palm formula (11.70) of the density function  $\varphi_1(\tau_1)$ .

Step 3. Using (11.36), the definition of dependencies:

$$\tau_1 = v_1^{-1}(u_1) \text{ and } \tau_j = v_j^{-1}(u_j), j > 1$$

and calculating the intervals between the moments of orders receipt.

Step 4. Calculating by the formula  $t_j = t_{j-1} + \tau_j$  of the flow elements  $\{t_j\}$ .

Step 5. Using the inverse function method, determining the dependency

$$D(t_j) = \Phi^1(u_j)$$

Step 6. Verifying  $t_j \leq IM$  ( $IM$  – modeling interval). If yes, go to step 2.

Step 7. Display results.

Step 8. End.

Table 11.1 – Formulas for modeling the main theoretical distributions of continuous random variables

Distribution	Density function	Modeling formula
Normal	$\varphi(\tau) = \frac{1}{\sigma_\tau \sqrt{2\pi}} \exp \frac{-(\tau - m_\tau)^2}{2\sigma_\tau^2}, -\infty < \tau < \infty$	$\tau = m_\tau + \sigma_\tau \left( \sum_{i=1}^{12} u_i - 6 \right)$
Uniform	$\varphi(\tau) = \frac{1}{b-a}, \tau \in [a, b]$	$\tau = a + u(b-a)$
Exponential	$\varphi(\tau) = \lambda e^{-\lambda\tau}, \tau \geq 0$	$\tau = -\frac{1}{\lambda} \ln u$

Distribution	Density function	Modeling formula
Linear	$\varphi(\tau) = \lambda \left(1 - \frac{\lambda}{2} \tau\right), \tau \in \left(0, \frac{2}{\lambda}\right]$	$\tau = -\frac{2}{\lambda} (1 - \sqrt{u})$
Gamma	$\varphi(\tau) = \frac{\alpha^k}{(k-1)!} \tau^{(k-1)} e^{-\alpha\tau}, \alpha > 0, k > 0, \tau \geq 0$	

Let consider the second case of demand modeling.

Demand is significantly affected by changes of product prices. A large influence on the level and dynamics of prices has the state of the financial and credit sphere, while the direct influence on prices is exerted by changes in the purchasing power of the country's monetary unit.

Let consider such a case of demand modeling, when demand is influenced by the price of products. Then dependence is legitimate:

$$D[t_j - t_i] = A + BP_i + U.$$

Here, the coefficients  $A$  and  $B$  are determined by standard econometric methods.  $U$  – a random variable characterizing fluctuations in preferences and incomes of consumers and other factors affecting the demand.  $P_i$  – the price of product on the  $i$ -th time interval.

Let create an algorithm for modeling the non-stationary Poisson income process using the example of a uniform distribution law of resource volumes [56].

Step 1. Setting initial data for modeling:  $T_0 - T_n$  – time interval;  $n$  – the number of sub-intervals;  $T[j]$  – the upper boundary of the  $j$ -th sub-interval;  $\lambda$  – the intensity of the process of each sub-interval ( $n$  times); parameters of the uniform distribution law  $a$  and  $b$

Step 2. Initial data set and organization of their storage in the database.

Step 3.  $t = 0, \lambda_{max} = \max \{\lambda[t]\}, k=1$ .

Step 4. Modeling random numbers  $u_1, u_2$  and  $u_3$  uniformly distributed on the interval  $[0, 1]$ .

Step 5. Calculating the value of the moment of arrival of the non-stationary Poisson process by the thinning method of Lewis and Shedler [18]:

$$t_j = t_{j-1} - \frac{1}{\lambda_{max}} \ln u_1.$$

Step 6. Organization of the cycle according to the number of sub-intervals ( $j$ ).

Step 7. Verifying  $t \leq (T(j) - T(j-1))$  if yes,  $\lambda(t) = \lambda(j)$ , go to step 10.

Step 8. Scaling up  $j = j + 1$ .

Step 9. Verifying the condition  $j \leq n$ , if yes, go to step 7. If no, go to step 13.

Step 10. Verifying  $\{u_2 \leq \lambda(t) / \lambda_{max} \text{ and } t \leq T_n\}$  if yes,  $t[k] = t, k = k + 1$ , go to step 12.

If no, then go to step 4.

Step 11. Calculating random value  $S=a+ u_3*(b-a)$ , obtained by the inverse function method.

Step 12. Saving the result of the experiment in the database, outputting the moment of occurrence of the event  $t_j$  and the value of the random variable  $S$ , go to step 4.

Step 13. End of the cycle.

Step 14. Preparation and printing the results.

### 11.6.2 Demand forecasting

Demand forecasting for manufactured products for a certain period of time in future is one of the important tasks of supply organizations. The type of forecast and the duration of the forecast time interval depends on the output and the behavior of demand for it. The most common type of forecast is based on statistical analysis methods. Here, in turn, one of the main methods is the least squares method. As a predictor, linear (11.72) and cyclic (11.73) functions are most often used [51]:

$$D_e = a + bt; \quad (11.72)$$

$$D_c = a + u \cos \frac{2\pi}{N} t + \mathcal{G} \sin \frac{2\pi}{N} t. \quad (11.73)$$

Expression (11.72) is typical for products for which the demand is growing to a certain extent, and expression (11.73) is typical for products with seasonal fluctuations in demand.

The creation of the forecasting function is shown by the example of a linear function (11.73). To do this, on the basis of the least squares method, it is necessary to minimize the sum of squared differences between the actual demand  $D$  at time  $t$  and the forecasting function:

$$e = \sum_{t=1}^n (D - a - bt)^2 \quad (11.74)$$

Further, equating the partial derivatives to zero, we obtain the equations:

$$\frac{\partial e}{\partial a} = \sum (D - a - bt) = 0; \quad (11.75)$$

$$\frac{\partial e}{\partial b} = \sum (D - a - bt)t = 0. \quad (11.76)$$

The determinant of the system of equations (11.74) - (11.76) has the form

$$\begin{vmatrix} D_e & 1 & t \\ \sum D & n & \sum t \\ \sum Et & \sum t & \sum t^2 \end{vmatrix}.$$

Then solving this system of equations we obtain:

$$a = \frac{\sum D \sum t^2 - \sum t \sum Dt}{n \sum t^2 - (\sum t)^2};$$

$$b = \frac{n \sum Dt - \sum D \sum t}{n \sum t^2 - (\sum t)^2}.$$

Introducing the assumption that  $t$  takes sequentially integer values from 1 to  $n$ , we have

$$\sum t = \frac{n(n+1)}{2};$$

$$\sum t^2 = \frac{n(n+1)(2n+1)}{6}$$

Consequently,

$$a = \frac{2(2n+1) \sum D - 6 \sum Dt}{n(n+1)};$$

$$b = \frac{6[2 \sum Dt - (n+1) \sum D]}{n(n^2-1)}.$$

Therefore, the procedure for demand forecasting is described on the basis of the method of statistical analysis of products for a certain period of time in future.

We create an algorithm for forecasting the value of the demand  $D$ .

Step 1. Initial data set.

Step 2. Cycle organization:  $i=1$ .

Step 3. Calculating

$$a = \frac{2(2n+1) \sum D - 6 \sum Dt}{n(n+1)}; \quad b = \frac{6[2 \sum Dt - (n+1) \sum D]}{n(n^2-1)}.$$

Step 4. Calculating the demand  $D_e$ .

$$D_e = a + bt$$

- Step 5. Cycle scaling up:  $i = i + 1$ .  
 Step 6. Verifying:  $i \leq n$ . If yes, go to step 3. if no, go to step 7.  
 Step 7. Display the results  $D_e$ .  
 Step 8. End.

### 11.6.3 Algorithm for the distribution of goods and products

The structure of the relationship of the algorithms for the distribution of goods and products (complex algorithm) is shown on the figure 3.4. Let describe the blocks shown on the figure 11.17. We start with the block 1, which simulates the demand  $D$  coming from consumers according to the algorithms given in chapter 11.4. In the second block, the values of  $R$  and  $S$  are compared, where  $R$  - the amount of goods and products available at the warehouse. In case of a shortage (block 3), when the demand exceeds the available resources, there are considered the distribution mechanisms for goods and products. The fourth block is an implementation of one of the selected distribution mechanism algorithms [25, 45].

If there is no deficit, in the fifth block it is checked whether the optimal distribution of goods and products is needed, if not necessary, then the natural decision of the warehouse is to satisfy the demand (block 9). If there is a need for the optimal distribution of  $R$ , there is a check for the presence of "disturbance" in the optimization problem in block 6. In case of the presence of "disturbance", the optimization problem is solved by the developed algorithm of the extension method presented in [54]. In case of the absence of "disturbance", the optimization problem is solved by standard optimization algorithms. Block 10 analyzes the influence of the changed parameters on the resulting solution and determines the range of changes in the model parameters, leading to an improvement in the optimal solution of the problem. This block in the structure plays a significant role both in interpretation and in ensuring the required reliability of the obtained modeling results. The results of computer modeling can be not only specific quantitative data on the values of the distributed goods and products, but also an analysis of the impact of various changes in non-stationary parameters on the efficiency of using algorithms for the distribution of goods and products.

The analysis of modern economic relations shows that the flows of resource inflows (materials, raw materials, orders, goods and products) are formalized in the form of non-stationary Poisson income processes with a wide range of random patterns characterizing volume indicators of resources.

During the distribution of resources, a number of problems arise related to both various economic priorities and the mathematical implementation of distributions. The generalized structure of the system for modeling and distribution of resource flows between objects includes the following steps:

- modeling of non-stationary Poisson processes of resource inflows;
- modeling and optimization of the distribution of resources volumes for each moment of resources receipt;
- analysis of the results and making adjustments to the initial data of the simulation model to obtain the desired results [46].

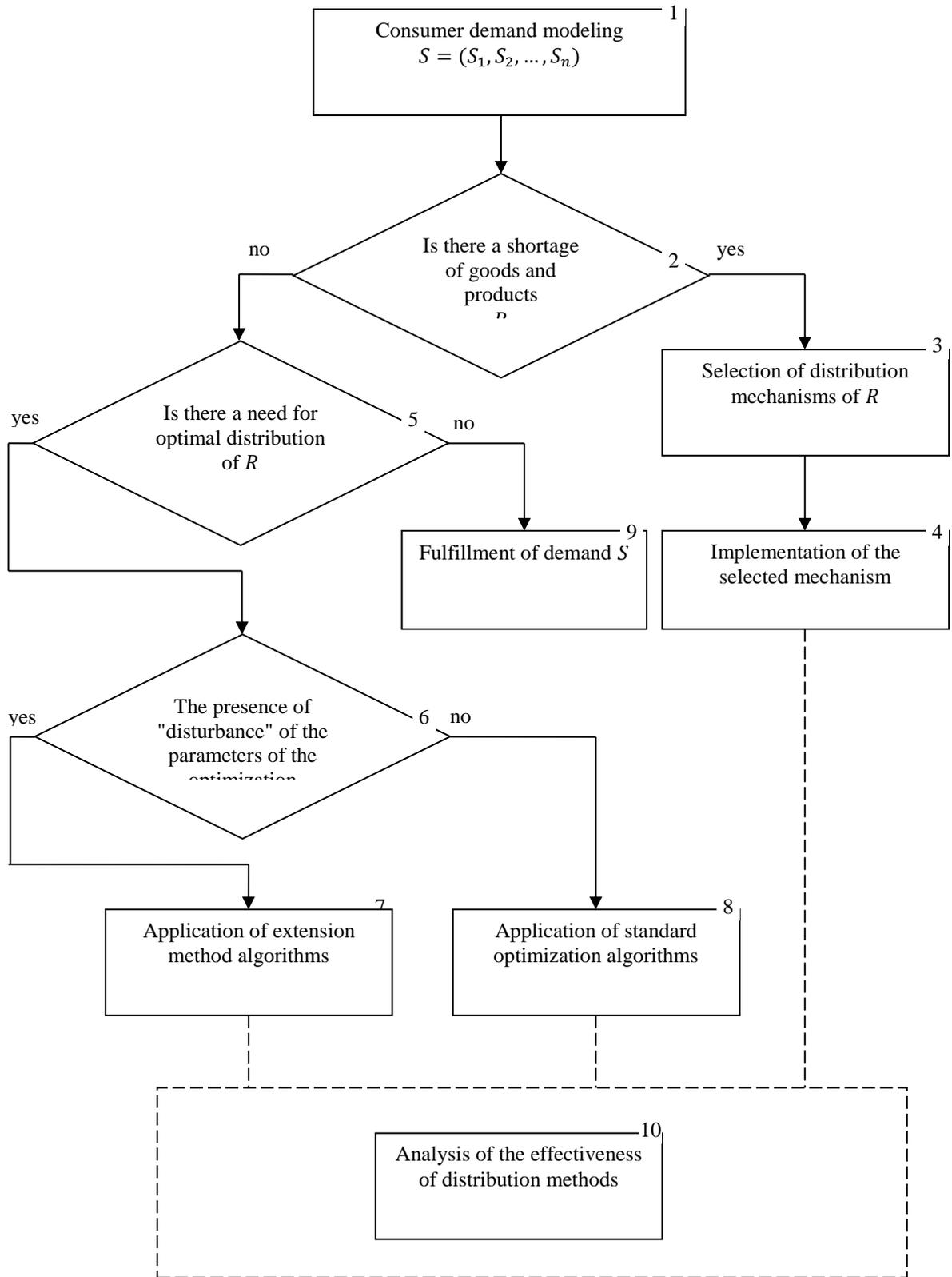


Figure 11.17 – The structure of the relationship of the algorithms for the distribution of goods and products

Below there are given algorithms for the distribution of goods and products.

Direct priority mechanism algorithm.

Step 1. Start

Step 2. Initial data set:

-  $n$  – number of consumers;

-  $R$  – amount of resources;

-  $\{r_1, r_2, \dots, r_n\}$  – optimal resource request for each consumer;

Step 3. Calculating the total amount of the requested resource:

$$sum = \sum_{i=1}^n r_i$$

Step 4. Verifying the condition  $sum \leq R$ , if yes, we satisfy the demand of each consumer:

$$x_i = r_i, \quad i = \overline{1, n},$$

if not, then

$$x_i = \frac{r_i}{sum} R, \quad i = \overline{1, n}.$$

Step 5. Calculating the maximum amount of the issued resource per consumer, for this we equal the customer's request to the total amount of the resource  $s_i = R$  and obtain:

$$X = \frac{R}{nR};$$

Step 6. Check which consumer has solved his problem if  $r_i \leq X, i = \overline{1, n}$  and calculate his request:

$$s_i^* = \frac{s_i}{s_i + (nR - R)} = X,$$

all other requests are left equal to  $R$ ;

Step 7. Calculating the selected amount of resources for each consumer:

$$x_i^* = \frac{s_i^*}{\sum_{j=1}^n s_j^*} R.$$

Step 8. Display the solution results.

Step 9. End.

It was proved [4] that this algorithm completes its work in no more than  $n$  steps.

The described resource distribution mechanism is perhaps the simplest one. Its meaning lies in the fact that all orders are proportionally “cut”.

Algorithm of inverse priorities mechanism.

Step 1. Start.

Step 2. Initial data set:

- $n$  – number of consumers;
- $R$  – amount of resources;
- the value  $A$ , characterizing the loss of an organizational system for each consumer  $A_1, \dots, A_n$ ;

Step 3. Determining the parameter of the balance constraint:

$$\gamma = \left( \frac{R}{\sum_{i=1}^n \sqrt{A_i}} \right)^2 ;$$

Step 4. Calculating equilibrium orders from the condition:

$$s_i^* = R \frac{\sqrt{A_i}}{\sum_{j=1}^n \sqrt{A_j}} ;$$

Step 5. Calculating the selected amount of resources for each consumer:

$$x_i^* = \min(s_i^*, \gamma(A_i / s_i))$$

Step 6. Display the obtained decision results.

Step 7. End.

With a sufficiently large number of consumers, the mechanism of inverse priorities with penalties for discrepancy between the expected and planned effects is optimal in the sense of total efficiency.

The inverse priority mechanism has several advantages. In particular, there is no unjustified overstatement of orders, i.e. no situation  $s_i^* > r_i$  arises. In addition, due to the condition of the reasonable behavior of Consumers (i.e., when each of them uses an equilibrium strategy  $s_i^*$ ), they receive as much as they ask. The disadvantage is that the numbers  $s_i^*$  are likely to be less than real needs  $r_i$ . As a result, the Warehouse does not receive reliable information about the real deficit)  $(\sum_{i=1}^n r_i) - R$ .

Algorithm of open control mechanism:

Step 1. Start.

Step 2. Initial data set.

- $n$  – number of consumers;
- $R$  – amount of resources;
- the value  $A$ , characterizing the loss of an organizational system for each consumer  $A_1, \dots, A_n$ ;

Step 3. Calculating prices for the resource:

$$h = \frac{\sum_{i=1}^n r_i - R}{\sum_{i=1}^n \frac{r_i^2}{2A_i}}.$$

Step 4. Calculating the distributed resource to consumer:

$$x_i = r_i \left( 1 - h * \frac{r_i}{2A_i} \right), \quad i = \overline{1, n}.$$

Step 5. Display the obtained decision results.

Step 6. End.

The described mechanism is an open control mechanism. Indeed, in the final analysis, all Consumers are divided into priority (which received as much as requested) and non-priority. Priorities receive as much as they ask, so it makes no sense to distort their real needs. Non-priority ones, as it is easy to see, cannot increase the resource selected to them by either raising or lowering their order. Thus, during the distribution of resources in accordance with the described mechanism, the Warehouse receives reliable information about the real needs of Consumers.

### Conclusion

This chapter discusses a set of dynamic models that characterize various levels of organization of economic systems and allow to identify the dynamics of development of credit and investment resources of an industrial enterprise in conditions of market uncertainty. For all the proposed models, there are developed effective algorithms that can be implemented in any programming language for modeling and analyzing the behavior of the model to change various economic parameters in order to develop optimal solutions.

Also there are developed decision-making methods for various priorities in the distribution of goods and products, in which the decision on how goods and products should be distributed among consumers is determined on the basis of their priority functions, the argument of which is consumer demand.

## Chapter 12. COMPLEX SYSTEMS MODELING

### 12.1 Modeling of an air attack on a ship

Let consider a simplified model of air attack of a missile carrier aircraft on a warship armed with ground-to-air missiles [17]. The purpose of the modeling is to assess the impact of the tactical and technical data of both parties on the result of an air attack.

#### 12.1.1 Air attack scheme and assumptions

An enemy aircraft with a full missile ammunition flies to the target (to the ship) at a constant speed and when the airspace limit is reached that corresponds to the threshold range of the aircraft missile (AM) flight, it launches the first and then subsequent missiles until their ammunition is exhausted or until that moment when it will be hit by a ship missile (SM). After the entire ammunition has been exhausted, the aircraft turns and leaves at the same speed.

In turn, the ship is characterized by a firing zone, depending on the detection range of the enemy aircraft. The detection range is assumed to be a random variable with a given distribution law. The maximum number of missiles that a ship can launch on an enemy's aircraft depends on the duration of the missile's stay within the firing zone and can reach the number of missiles in the ship's ammunition. To simplify the air attack reflection model, we will accept the following assumptions:

1) The object of destruction for the aircraft is the ship, and for the ship - the enemy's aircraft, and not missiles launched by it.

2) Both the aircraft and the ship launch the missile only when the enemy enters the corresponding firing zone.

3) As a result of an aircraft attack, the ship can either be damaged, or sunk, or not damaged, in turn, the aircraft either survives or can be shot down. Partial damage to the aircraft is not considered. Therefore, six outcomes are possible:

- the aircraft dies, the ship is not damaged;
- the aircraft dies, the ship is damaged;
- both sides perish;
- the aircraft is not damaged, the ship is damaged;
- the aircraft is not damaged, the ship is sunk;
- both sides are intact.

4) The speed of the ship is negligible compared to the speed of the aircraft.

5) A damaged ship does not lose the ability to retaliate.

#### 12.1.2 Parameters and equations of the air attack model

The model of an air attack on a ship is characterized by the following parameters:

$V$  and  $O$  – respectively, the speed of an aircraft and a ship;

$\Delta_c$  and  $\Delta_k$  – intervals between consecutive moments of the launch of the AM and SM;

$U$  and  $W$  – speeds of AM and SM;

$M$  and  $N$  – amount of AM and SM in enemy ammunition;

$S_k$  – enemy aircraft detection range;

$P_1(S_i)$ ,  $P_2(S_i)$ ,  $P_3(S_i)$  – respectively, the probability that the ship is not damaged, damaged, destroyed by the  $i$ -th AM;

$P_k(S_j)$  – probability of aircraft hitting by the  $i$ -th SM.

Describing the dependencies between the entered parameters, there is used relative time, i.e. the moment of launching the first missile by any side is taken as the time reference [34]. Consequently, the form of all the equations of the model will depend on which firing zone is larger for a particular implementation of the simulation model.

Taking into account the above remark and the assumptions made, we present the dependencies for determining:

– the moment of launching the last missile by the aircraft:

$$t_c = (M-1)\Delta_c \text{ at } S_c \geq S_k,$$

$$t_c = (S_k - S_c)/V + (M-1)\Delta_c \text{ at } S_c < S_k;$$

– the distance between aircraft and ship at any given time:

$$S(t) = S_c - Vt \text{ at } t < t_c, S_c \geq S_k,$$

$$S(t) = S_c - 2Vt_c + Vt \text{ at } t \geq t_c, S_c \geq S_k,$$

$$S(t) = S_k - Vt \text{ at } t < t_c, S_k > S_c,$$

$$S(t) = S_k - 2Vt_c + Vt \text{ at } t \geq t_c, S_k > S_c;$$

– the duration of the aircraft in the firing zone of the ship:

$$T = 2[(S_k - S_c)/V + (M-1)\Delta_c] \text{ at } S_k > S_c,$$

$$T = 2[(M-1)\Delta_c - (S_c - S_k)/V] \text{ at } S_k \leq S_c;$$

– the maximum number of missiles a ship can launch:

$$R = \lfloor T/\Delta k \rfloor + 1;$$

– the moment  $t_i$  of the  $i$ -th AM launch by the aircraft:

$$t_i=(i-1)\Delta c \text{ at } S_c \geq S_k,$$

$$t_i=(S_k-S_c)/V+(i-1)\Delta c \text{ at } S_c < S_k;$$

- the moments  $\theta_i$  of the target achievement by the  $i$ -th AM:

$$\theta_i=t_i+S(t_i)/(U+V);$$

- the moments  $\tau_j$  of the  $j$ -th SM launch:

$$\tau_j=(S_c-S_k)/V+(j-1)\Delta k \text{ at } S_c \geq S_k,$$

$$\tau_j=(j-1)\Delta k \text{ at } S_c < S_k;$$

- the moments  $\eta_j$  of the target achievement by the  $j$ -th SM.

Calculating the moments  $\eta_j$  the following cases must be considered.

Case 1. At the moment of the  $j$ -th SM launch, the aircraft goes to the target, i.e.  $\tau_j < t_c$ , and then  $\eta_j$  is a solution to the equation:

$$\eta_j - \tau_j - S(\eta_j)/(W+V) = 0.$$

Case 2. At the moment of the  $j$ -th SM launch, the aircraft moves away from the target, i.e.  $\tau_j > t_c$ , and then we solve the equation:

$$\eta_j - \eta_j - S(\eta_j)/(W-V) = 0.$$

### 12.1.3 Modeling algorithm creation

The functioning (duration of the attack) of the modeling process is considered in the interval from the moment the first missile of either side is launched until one of the six attack outcomes listed above occurs.

The modeling algorithm is created on the principle of sequential posting of orders. The block diagram of the algorithm is shown on the figure 12.1. Operator 1 enters the following initial information: the tactical and technical data of the aircraft, ship and their missile ammunition, the parameters of the distribution law of the length of the firing zone of the ship, which coincides with the detection range of the air target and is a random value, the initial value of the moment of the aircraft crash (MAC), taken obviously large. Operator 2, together with operators 53, 55, provides a given number of runs of the air attack model. With the help of the operator 3, the next implementation of the enemy aircraft detection range,  $s_k$ , is formed.

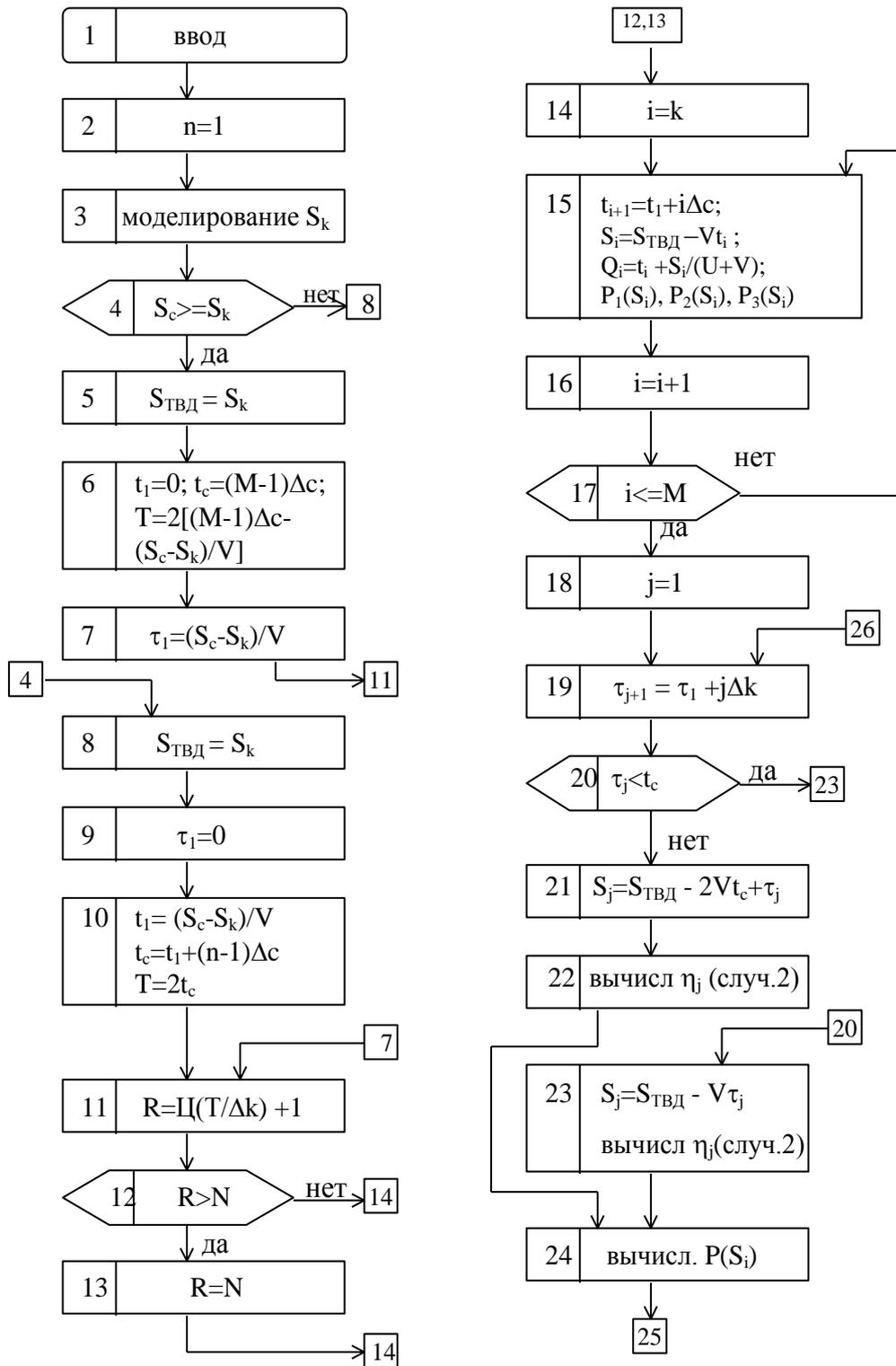
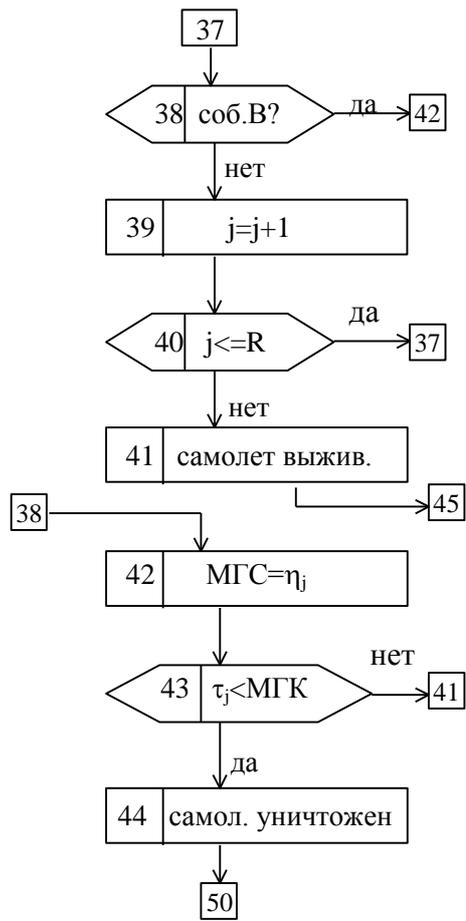
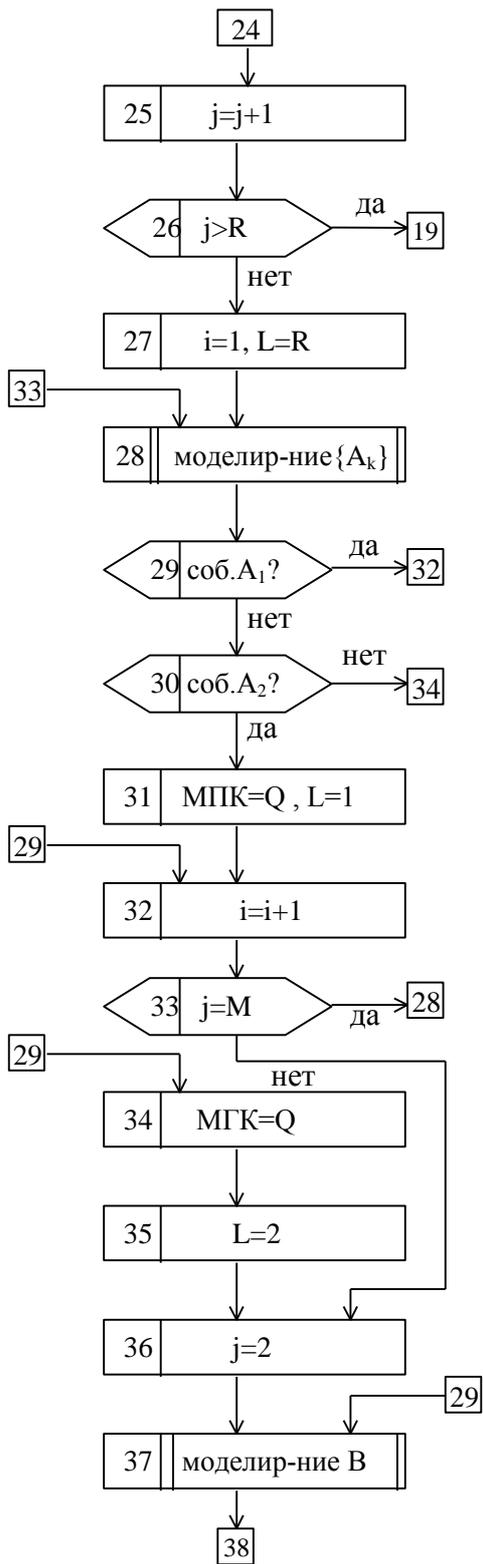
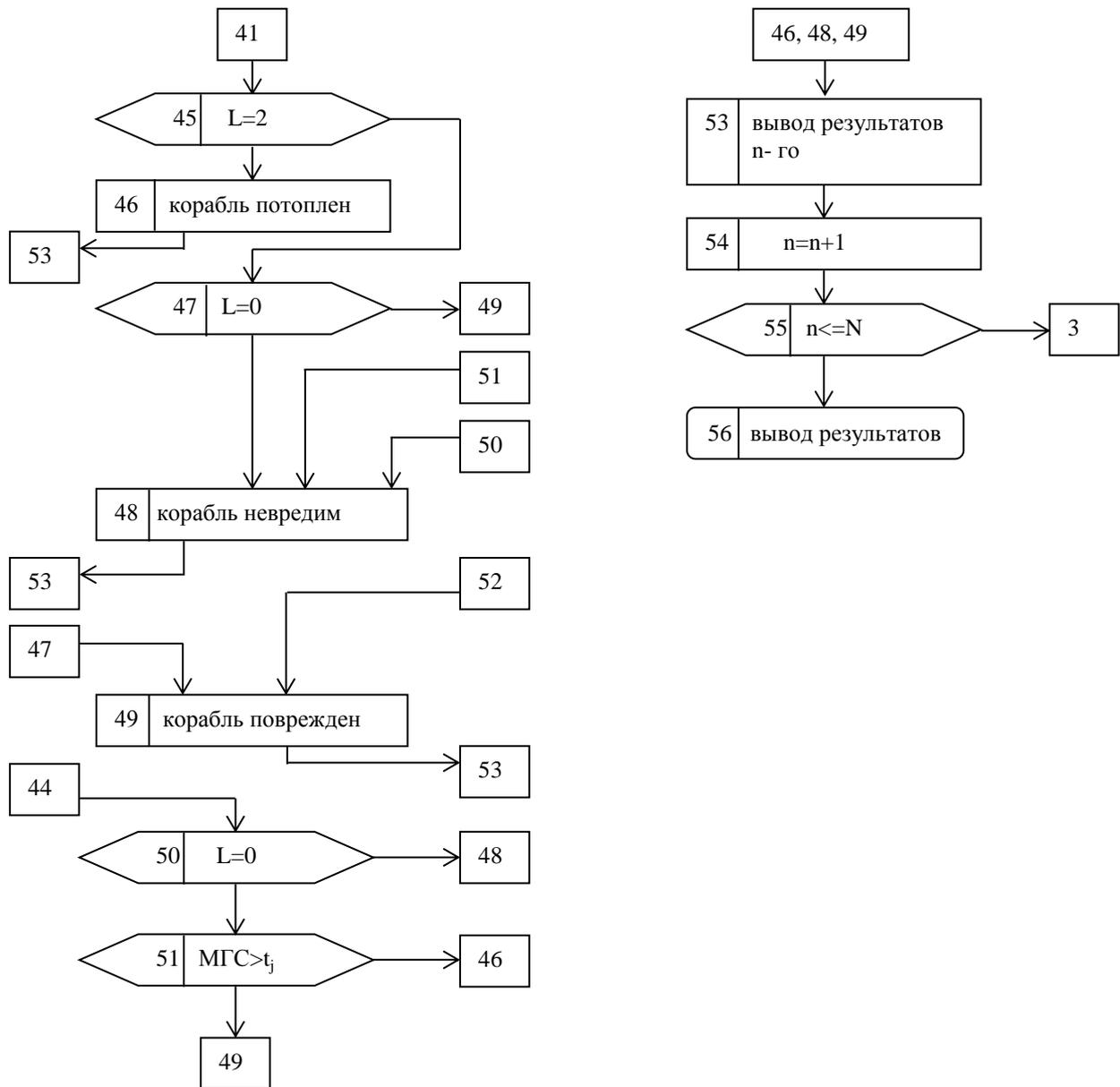


Figure12.1 - The block diagram of the modeling algorithm



Continuation of the figure 12.1



Continuation of the figure 12.1

Let introduce the sign  $\Phi$ , which takes the value 0, if the ship starts the battle first, and 1 if - the aircraft. If  $S_k \leq S_c$  (operator 4), then the aircraft begins the battle by launching its first missile. This fact is recorded by the operator 5, and the theater of battle operations is determined by the expression  $S_m = S_c$ , since the origin is the moment the first AM was launched (operator 6). Operator 6 also determines the launch time of the last AM and the duration of the aircraft in the firing zone of the ship. The values  $t_l, t_c, T$  and  $S_m$  for the case  $S_k > S_c$ , i.e. when the first missile launched is SM, it is determined by operators 8, 9, 10. Operators 7 and 9 determine the launch moments of the first SM. The number of missiles that the ship can launch during the aircraft's stay in the firing zone is found with the help of operators 11, 12, 13.

The group of operators from 14 to 17 provides calculations for each missile from the ammunition of the aircraft, the launch times  $t_i$ , the distance between the aircraft and the ship at the time of launch -  $S_i$ , the moment of defeat of this AM of

the ship  $\theta_i$  and the probabilities  $P_1(S_i)$ ,  $P_2(S_i)$ ,  $P_3(S_i)$  of the full group of events  $A_1$ ,  $A_2$ ,  $A_3$ . Here,  $A_1$  is the event that the ship will not be damaged as a result of the attack of the  $i$ -th AM,  $A_2$  - the ship is damaged,  $A_3$  - the ship is sunk.

The group of operators from 18 to 26 allows to find: the moments  $\tau_j$  of the SM launch (operator 19) and depending on whether the aircraft flies to the target or from the target (operator 20), the distance between the ship and the aircraft at these points of time (operators 21 and 23), the moments of the defeat of the aircraft by SM (operator 22 and 23), as well as the probability of the defeat of the aircraft by the  $j$ -th missile of SM (operator 24).

The next group of operators 27 to 33 determines the degree of damage to the ship by the  $i$ -th AM. Operator 28 models a complete group of events:  $A_1$ ,  $A_2$ ,  $A_3$ . If an event  $A_1$  occurs (the ship remains intact), then with the help of the operators 32 and 33 the complete group of events is simulated again.

Depending on the occurrence or non-occurrence of an event  $A_2$  (operator 30), operators 31 and 34 form the moments of damage or defeat of the ship. In order to recognize these events, each of them corresponds to an attribute  $L$  with values 1 and 2, respectively.

The group of operators from 36 to 44 works similarly. But here the outcome is survival (operator 41) or the possible defeat of an enemy aircraft (operator 42). The final determination of the fact of the defeat of the aircraft is carried out by the operator 44, if the conditions of the operator 43 are met. If the conditions of this operator are not fulfilled, launch of the SM at the moment  $t_j$  with a damaging outcome is impossible, since by this time the ship should already be defeated. Similarly, the battle outcome for a ship is set by operators 45 to 52.

Operators 53, 55 are service ones and provide a repetition of a given number of model implementation of the air attack on the ship.

The created modeling algorithm allows to assess the impact of changes in the tactical and technical data of both sides on the outcome of the battle.

## **12.2 Modeling the process of electricity distribution in the country**

### **12.2.1 Modeling object**

The task of efficient electricity consumption from year to year is becoming an urgent problem for all of humanity. This issue is also acute in Kazakhstan. The exceptional importance of electricity for all sectors of the economy necessitates the development of energy balances taking into account the optimal use of energy resources and the work of the fuel and energy economy of the country.

The functional diagram of energy distribution within the framework of the energy balance in the country is shown on the figure 12.2. The balance of electricity in Kazakhstan is characterized by the dynamism of the resource part and consumption volumes and at the same time weak structural shifts in the sources of electricity generation [47]. In this regard, for optimal distribution of energy resources, it is not enough to use only optimization methods, but it is necessary to develop a simulation distribution system that allows to take into account both the

dynamism of the object of study and possible fluctuations during the process of generation and consumption of electricity.

### 12.2.2 Simulation model structure

It should be determined by the nature and interconnection of the tasks that this system implements to achieve the goal. The purpose of developing a simulation system is the optimal distribution of energy resources within the framework of the energy balance in the country in conditions that are closest to the real ones. In order to do this, the simulation system must ensure the implementation of the following interrelated tasks:

- calculation of the regulatory demand for electricity for a given period of time by industry;
- identification of patterns of changes in the actual consumption of energy resources;
- simulation of fluctuations in the resource (input) part of the energy balance and actual electricity consumption;
- optimal distribution of limited energy resources between consumers;
- analysis and assessment of the influence of various factors on the efficiency of electricity use.

An analysis of the nature of the relationship between these tasks determines the following general structure of a simulation system for the distribution of energy resources, shown on the figure 12.3.

Let briefly dwell on each of the blocks of this structure.

### 12.2.3 Determination of power consumption by industry

Calculation of electricity consumption by sectors of the national economy is preceded by the task of forming the average rates of electricity consumption. For this, the energy demand for the production of the  $i$ -th type of product for the reporting, current and planned periods is calculated for each industry and department. Then, for all sectors, the total demand for electricity for the production of the  $i$ -th type of marketable products ( $\Pi_i$ ) and its total production volume ( $P_i$ ) are determined and the average rates of energy consumption for the production of the  $i$ -th type of product  $N_i = \Pi_i / P_i$  are easily calculated .

Based on the average consumption rates, there are carried out preliminary calculations in order to determine the need for electricity by energy systems, energy units and regions of the country in the context of industries and energy-intensive types of products. Calculations are performed for all planning modes: annual, five-year and long-term. All specific formulas for such calculations can be found in the specialized literature.

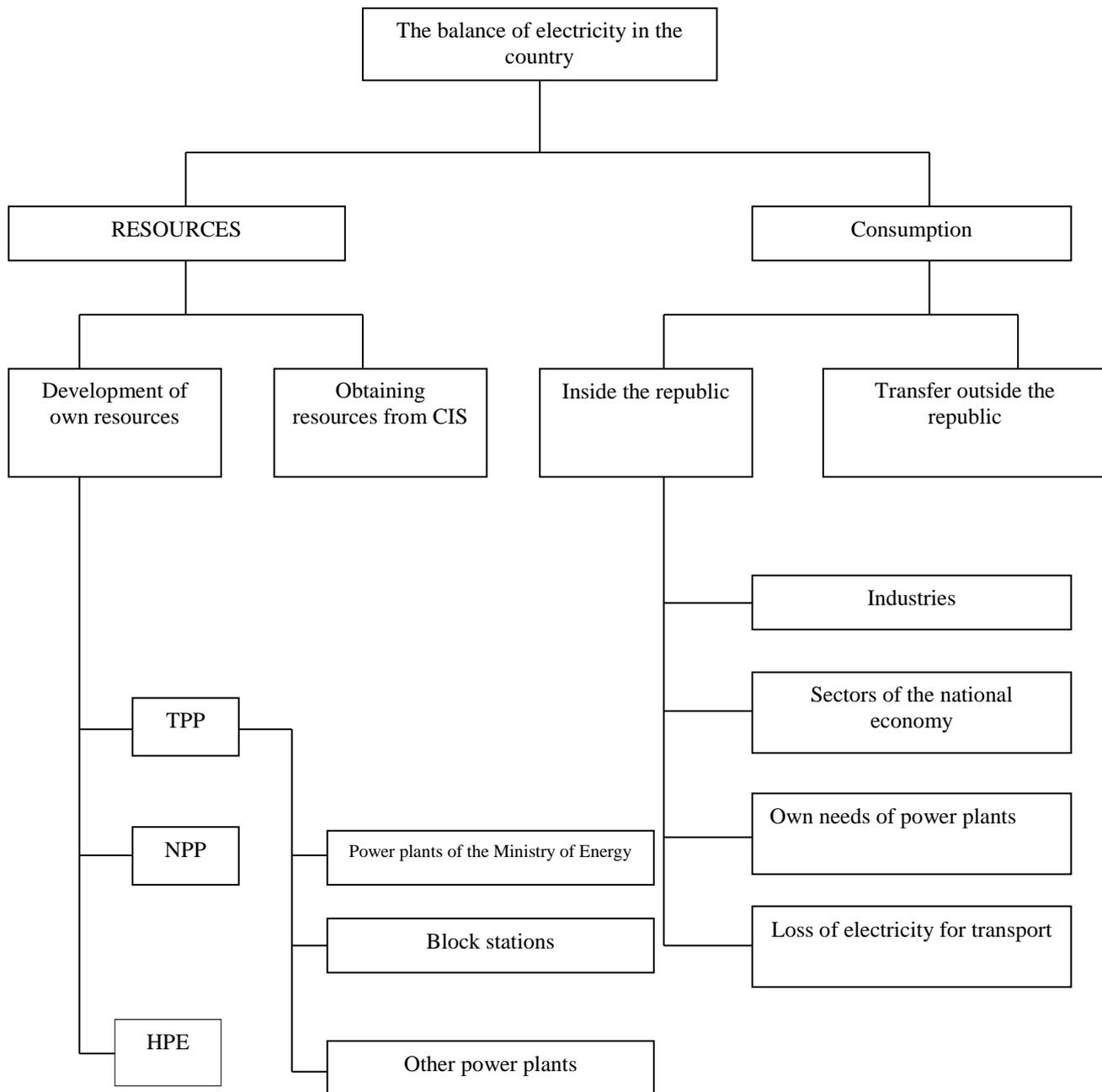


Figure 12.2 - Functional diagram of energy distribution

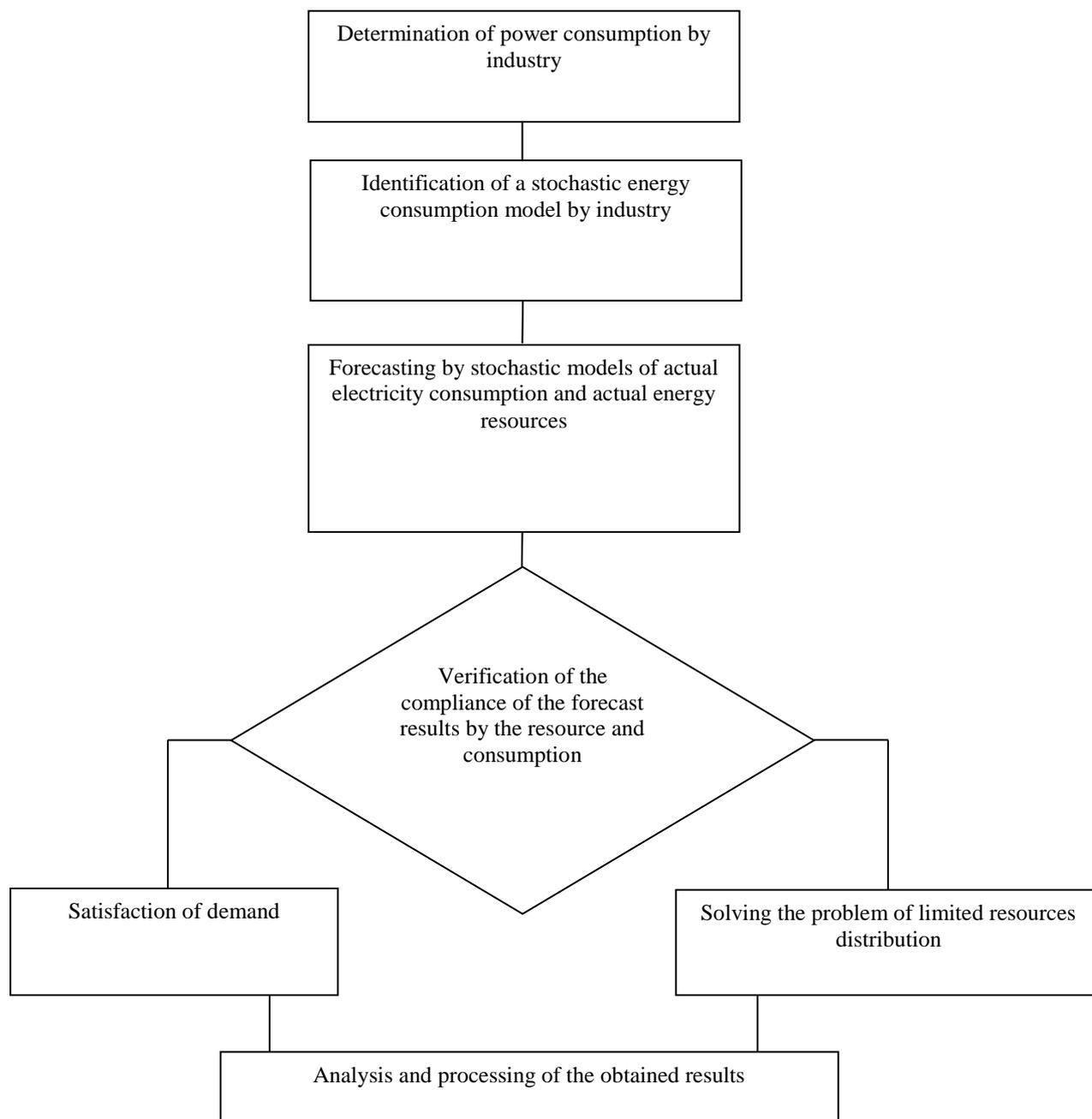


Figure 12.3 - Power Distribution IS Structure

#### 12.2.4 Identification of a stochastic model of energy consumption and production

The random nature of the volumes of consumption and production of electricity is an indisputable fact and it is impossible to neglect it during any research in this area. Therefore, the definition (identification) of the random variables distribution laws characterizing the volumes of consumption and production of electricity by energy systems and sectors of the national economy is one of the essential tasks of the considered simulation system. In this case, it is

necessary to use well-known methods and identification algorithms, for example, in [44].

Forecasting according to the obtained distribution laws of actual volumes of production and consumption of electricity is carried out using the appropriate algorithms for random patterns modeling given in chapters 2-7.

Blocks "Verification of the compliance of the forecast results by the resource and consumption", "Satisfaction of demand" do not require special consideration because of the simplicity of the implementation.

The problem of limited energy resources distribution is usually formed in the form of mathematical programming tasks consisting of an objective function and constraints. The criterion of the efficiency of electricity consumption in the country as a whole is used as the objective function. And the constraints of the task will be:

- total volume ( $B$ ) of production of marketable products;
- limited amount of electricity ( $L$ );
- characteristic features of individual electricity consumers related to the profitability and production development rate.

Considering the foregoing, the mathematical model of the energy distribution task can be written in the following form.

The block "Analysis and processing of results" in the simulation model plays

$$F = \max \sum_{j=1}^n (1/p_j)x_j;$$

$$\sum_{j=1}^n (1/N_j)x_j \geq B;$$

$$\sum_{j=1}^n x_j \leq L;$$

$$d_j \leq x_j \leq D_j, j = \overline{1, n}.$$

a significant role both in interpretation and in ensuring the required reliability of computer modeling results. Therefore, it is supposed to use the apparatus for experiment planning, discussed in Chapter 8.

Obviously, the simulation modeling results can be not only specific quantitative data on the values of electricity distributed to consumers, but also an analysis of the influence of various factors (unfavorable modes of electricity consumption, fuel shortages at power plants, etc.) on the efficiency of electricity use in the country.

## Chapter 13. COMPUTER MODELING, ANALYSIS AND OPTIMIZATION OF INVESTMENT PROCESSES

Computer modeling is one of the effective methods of analyzing economic systems. With the intensive development of computer technology and software, the scope of computer modeling in the economic sphere has expanded significantly. Let consider the main advantages of using computer modeling in investment activities.

The use of computer modeling methods allows to obtain a system of both quantitative and qualitative indicators characterizing the financial and economic assessment of investment activity, on the basis of which there are formulated practical recommendations on improvement of credit policy for medium and long-term purposes. To unstable random parameters and processes modeling of investment activity, there is used a modern apparatus of the theory of simulation modeling, which allows to simulate all classes of random economic and financial parameters and indicators with the instability factor specified by analytical laws.

### 13.1 Object of the research

In most countries of the world, national development institutions play an important role in ensuring the rate and dynamics of economic growth, and during the crisis play a stabilizing role. In the Republic of Kazakhstan, this mission is carried out by the Development Bank of Kazakhstan (DBK). The broader meaning of the DBK mission is to help to achieve sustainable long-term economic growth of the Republic of Kazakhstan by meeting on a commercial basis the investment needs of the accelerated development of competitive non-raw-materials sectors of the country's economy in credit resources not serviced by the second-level banks (Figure 13.1) [62].

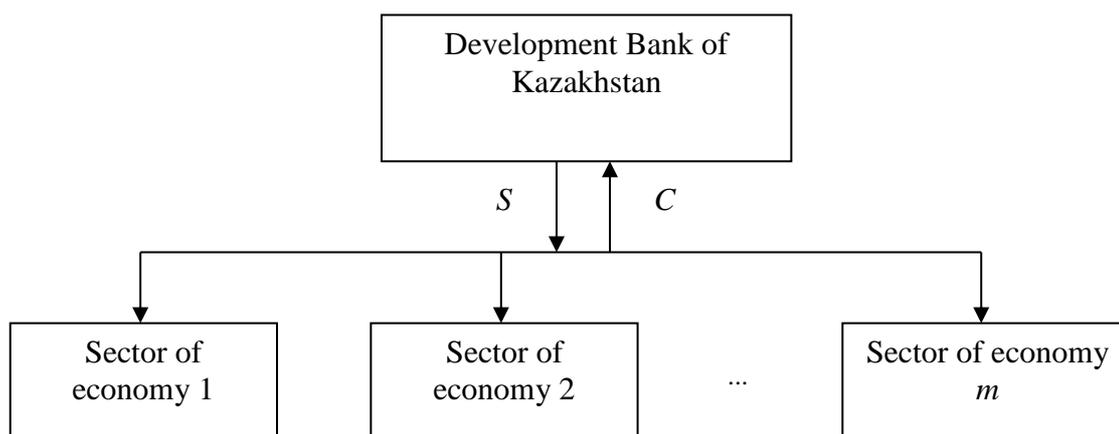


Figure 13.1 – The general scheme of the object of research

## 13.2 Functional diagram of the implementation of the development bank investment program

The Bank's structural divisions ensure the implementation of the basic procedures [8] related to the preparation for the implementation of the investment program (Figure 13.2).

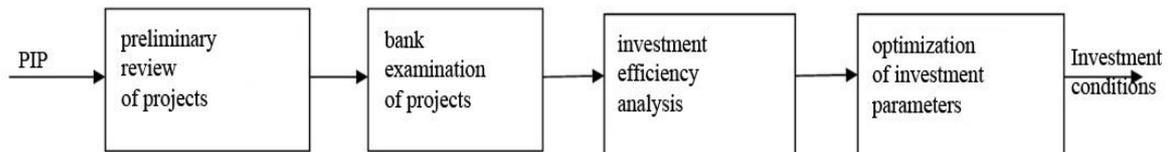


Figure 13.2 – The scheme of the pre-investment process

The first of these procedures involves consideration and preliminary examination of business proposals (BP) of borrowers.

The second procedure usually involves: assessing the current and future state of the enterprise, determining the possible rates of its development, assessing the possibility of mobilizing available sources of funds and determining the position of the enterprise in the capital market.

The following procedure, based on forecasting and calculations of the main financial indicators of projects (net present value, cash flows, discount factors, etc.), analyzes the preliminary investment efficiency.

The fourth procedure is carried out on the basis of finding the optimal investment parameters.

Let consider in more detail each of the procedures.

### 13.3 Preliminary projects review

#### 13.3.1 Stages of preliminary examination of business proposals

The preliminary review of projects is carried out in 2 stages:

- 1) attracting and advising potential Borrowers regarding the Bank's Credit Instruments with the aim of preliminary determination of financing variants;
- 2) consideration of the Project for the possibility of their financing by the Bank, taking into account the requirements of the legislation of the Republic of Kazakhstan, internal documents of the Bank, as well as attractiveness of BP for the Bank under the Project.

Attracting and advising potential Borrowers can be represented in the form of the following scheme (Figure 13.3).

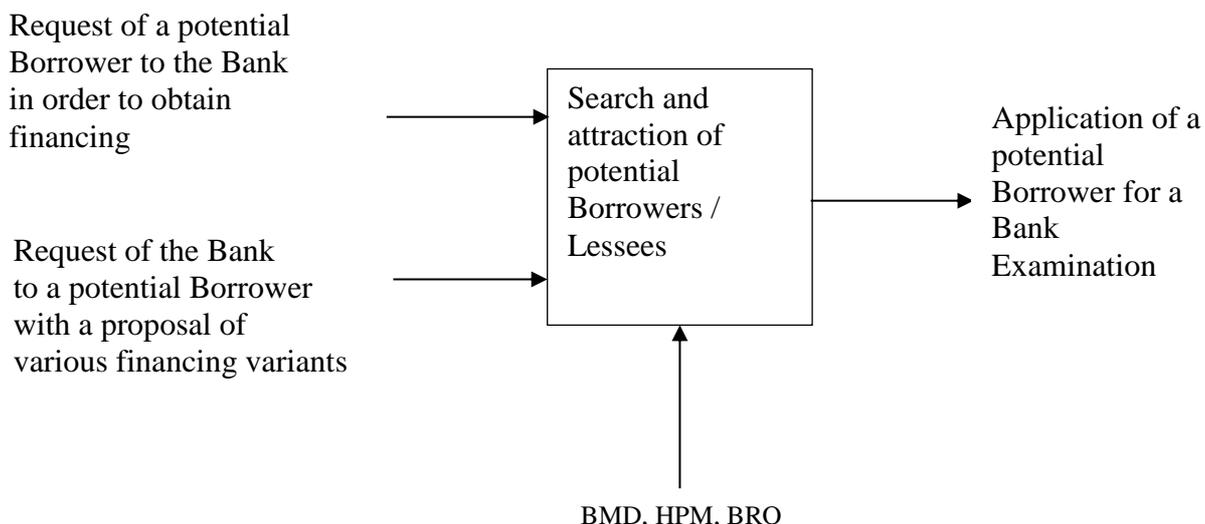


Figure 13.3 – Scheme of search and attraction of potential Borrowers/Lessees

The figure 13.3 uses the following abbreviations:

*BMD* – Bank Managing Director;

*HPM* – Head of Project Management;

*BRO* – Bank Representative Office in Almaty, carrying out the organization of procedures at the stage of searching and attracting potential Borrowers, at the stage of preliminary consideration of BP, at the stage of monitoring the Project of Almaty and Almaty region, as well as carrying out comprehensive monitoring of the Alarm assets of Almaty and Almaty region in accordance with current Regulations.

The second stage of the preliminary project review procedure will be presented in the form of a diagram presented on the figure 13.4, a step-by-step algorithm of which is given below.

The figure 13.4 uses the following abbreviations:

*AD* (Administrative Department) - for registration of BP;

*PD* (Project Directorate) - for the implementation of a qualitative review of BP, the provision of a complete package of documents of the Bank SS and the preparation of indicative financing conditions;

*Deputy Chairman of the Management Board/Credit Committee of the Bank* - for making a decision on BP;

*DOE* (Department of Collateral Evaluation) - for conducting a preliminary analysis of the collateral structure;

*DCrR* (Department of Credit Risks) - for timely consideration and approval of indicative financing conditions;

*DSI* (Department of Strategy and Information) - for registration of BP;

*SS* (Security Service) - for providing a conclusion on the legal capacity of the Counterparties, the presence/absence of negative information regarding the Counterparties, as well as a list of legal entities and individuals affiliated with it;

*CS* (Compliance Service) - for the implementation of the procedure of counteraction to the legalization (laundering) of illegally gained income and the

financing of terrorism in accordance with the Bank's internal acts and the requirements of the legislation of the Republic of Kazakhstan.

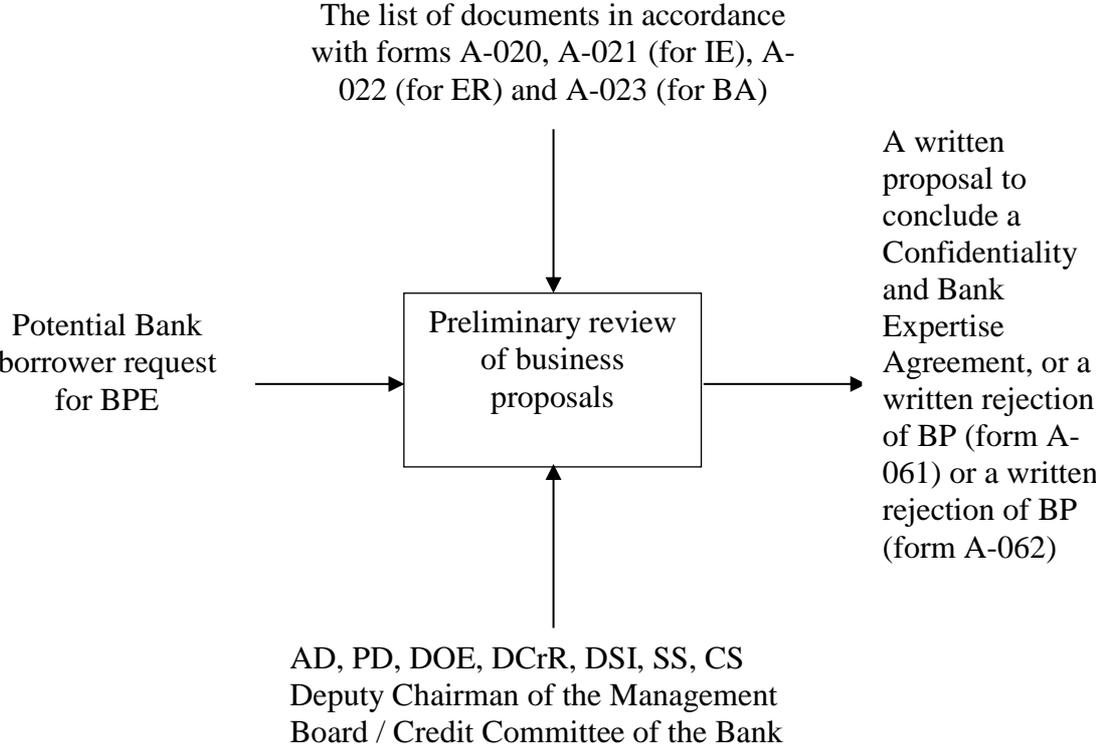


Figure 13.4 – Business proposal preliminary review scheme

13.3.2 Simulation model of preliminary analysis of a package of documents of a borrower

Let the bank receive packages of documents of borrowers ( $j$  - the number of the received request), which form an ordinary flow of homogeneous events with the specified distribution of intervals between events  $\tau$ , requests are lined up and served in the order of receipt, and we will consider the bank as a MSS with one service channel.

Upon receipt of requests by the bank, they are lined up and served in the order received.

The duration of the request service is a random variable, depending on the following components:

- $t_j$  – time of receipt of a package of documents of the borrower to the Deputy Chairman of the Management Board;
- $\tau$  – time interval after which the next package of documents arrives at the bank;
- $\tau_{3mn}$  – time required for the Deputy Chairman of the Management Board to review documents;

$\tau_{3nn}^{cb}$  – the time of exemption from the  $j$ -th request by the Deputy Chairman of the Management Board in charge of PD (project management), no more than 1 business day;

$\tau_{y\delta}$  – time required for consideration of the package of documents by the Managing Director in charge of the PD;

$\tau_{y\delta}^{cb}$  – transmission time of the package of documents of the  $j$ -th request to the PD, no more than 1 business day;

$\tau_{c\delta}$  – time required to verify the details of the applicant's form of the  $j$ -th request, no more than 3 days;

$\tau_{c\delta}^{cb}$  – moment of time when the security service (SS) transmits an opinion with conclusions on this package of documents;

$\tau_{ck}$  – time during which the CS checks the ownership of the Applicant ( $j$ -th request), the ultimate beneficial owner, affiliates to a foreign official, and also assesses his reputation, information of the applicant's profile, no more than 2 days;

$\tau_{ck}^{cb}$  – moment of time when the CS gives its opinion;

$\tau_{ah}^{cb}$  – time required for analysis of counterparties and preliminary consideration of the project content for compliance with the current priorities of the bank on the  $j$ -th request, no more than 5 days;

$\tau_{\delta ka}^{cb}$  – time during which the Department of Credit Administration draws up a report on the  $j$ -th request, no more than 1 business day;

$\tau_{k\delta}^{cb}$  – time required to verify the location of the business entity and the proposed pledge of the  $j$ -th request, no more than 3 days.

Accordingly, the service time of the  $j$ -th request is the sum of the previously considered variables:

$$\tau_j^{o\delta cl} = \tau_j^h + \tau_{3nn}^{cb} + \tau_{y\delta}^{cb} + \tau_{c\delta}^{cb} + \tau_{ck}^{cb} + \tau_{ah}^{cb} + \tau_{\delta ka}^{cb} + \tau_{k\delta}^{cb},$$

where  $\tau_j^h$  – start of  $j$ -th request service.

An algorithm that implements the process of preliminary analysis of a package of documents can be represented in the following form [26].

*Step 1.* Receiving a package of documents on BP of the  $j$ -th applicant.

*Step 2.* Performing a test to end the specified simulation interval. If  $t_j > T$ , then go to step 13.

*Step 3.* Modeling  $\tau$  (modeling time period after which the next request to the bank will be filed).

*Step 4.* Consideration of a package of documents within 1 business day, assignment of the corresponding values of  $R_l$  and  $K_l$  and sending documents to the managing director (MD) in charge of the PD, where  $R_l$  – a feature of the reliability of the submitted documents, and  $K_l$  – a feature of the full volume of the submitted documents.

*Step 5.* MD sends a package of documents to PD within one working day.

*Step 6.* PD checks the package of documents for completeness ( $K_1$ ) and the compliance of the electronic package of documents of the project with the original ( $R_1$ ). If both conditions ( $K_1=1$ ,  $R_1=1$ ) are satisfied, then go to step 8.

*Step 7.* Sending a request for additional documents to the applicant with a corresponding letter.

*Step 8.* Sending documents to the security service (SS) and the compliance service (CS) for appropriate verification and notification to the DSM (Department of Strategy and Management) for registering of BP.

*Step 9.* Registration of DSI BP.

*Step 10.* ISS verification of the applicant's questionnaire (within three business days):

- identifies the provision of false information ( $R_2$ );
- reveals participation in dishonest, financial fraud and etc. ( $R_3$ );
- establishes the affiliation of the applicant with the project participants, bank officials ( $R_4$ );
- takes measures to establish the beneficial owner of the applicant ( $R_5$ ).

If  $R_2=1$ ,  $R_3=1$ ,  $R_4=1$ ,  $R_5=1$  then go to step 11, if no – go to step 13.

*Step 11.* The SS sends a conclusion about the price of property to establish business relations with the applicant to the Deputy Chairman of the Management Board (in charge of PD) and a copy of the CS. The Deputy Chairman of the Management Board shall forward the opinion of the SS to the respective MD.

*Step 12.* CS within 2 business days performs:

- verification of affiliation of the applicant, the ultimate beneficial owner of affiliates to a foreign official (FO) ( $L_1$ );
- assessment of the applicant's reputation (FO), etc., in relation to involvement in cases of legalization (laundering) of illegally gained income and financial terrorism ( $L_2$ ).

Submissions of the results of the verification to MD.

If  $L_1$  and  $L_2$  are not equal to zero, then go to step 14.

*Step 13.* Deputy Chairman of the Management Board removes BP from consideration, go to step 24.

*Step 14.* BP is sent for analysis of counterparties and preliminary consideration of the project contents for compliance with the current priority of the bank ( $Pr=1$  yes,  $Pr=0$  no) within five working days.

*Step 15.* Service letter of non-compliance.

*Step 16.* CAD (Credit Administration Department) sends a report to a credit bureau within one business day.

*Step 17.* Checking the location of the business entity and the proposed collateral within three business days ( $Ad=1$ , when the presented data coincides with the real one).

*Step 18.* Verification: credit history is clean ( $KrdH=1$  – clean), if not => step 20.

*Step 19.* Collateral availability check ( $Zl=1 \rightarrow$  no collateral), if  $Zl=0 \Rightarrow$  step 13.

*Step 20.* Decision of the Deputy Chairman of the Management Board on the rejection of BP from consideration. If the solution is positive, then go to step 13.

*Step 21.* Preparation of indicative financing conditions (IFC).

*Step 22.* IFC Credit Risk Review and Approval by the Department.

*Step 23.* Verification: rejection of BP or approval of IFC? If rejected, go to step 13.

*Step 24.* Sending a letter to the applicant.

*Step 25.* Calculation of modeling parameters and statistics.

*Step 26.* Sending corresponding notification to DSI.

*Step 27.* The end of the preliminary analysis of the package of documents of BP.

In order to calculate statistics, there were adopted following notations:

*Sr1* – the number of BP in which false data were submitted;

*Sr2* – counter of BP with false information;

*Sr3* – counter of BP, whose applicants were convicted of participation in fraudulent transactions, financial fraud, etc.;

*Sr4* – number of BP whose applicants are affiliated with project participants, bank officials;

*Spol* – the number of BP accepted for examination;

*Sotk* – the number of BP rejected for examination.

After a preliminary review of projects, the Bank proceeds to the following procedure of the BPE Implementation investment process, as a result of which there is made an assessment of the current and future state of the enterprise, the determination of the possible rates of the enterprise development, the assessment of the possibility of mobilizing available sources of funds and determining the position of the enterprise on the capital market.

### **13.4 Assessment of the status and development prospects of the borrower**

Assessment of the financial position of the enterprise is based on data from its balance sheets for the previous period, as well as other reporting technical and economic documentation. Below there is a brief list of generalized financial indicators that are usually used for such an assessment, divided into four groups [28, 53].

*I. Liquidity ratios (used to assess the ability of the company to fulfill its short-term liabilities).*

Liquidity - the ability of assets to be quickly sold at a price close to the market one. Liquidity is the ability to turn into money.

Usually there are distinguished highly liquid, low liquid and illiquid values (assets). The easier and faster you can get the full value of an asset, the more liquid it is. For a commodity, liquidity will correspond to the rate of its sale at a nominal price.

### *Calculation of liquidity ratios*

In practice of financial analysis, there are three main indicators of liquidity.

– *current liquidity ratio (current ratio, CR)* - a financial coefficient equal to the ratio of current assets and current liabilities of the company.

It is calculated according to the company's balance sheet as the quotient of dividing current assets by current liabilities and shows whether the enterprise has enough funds that can be used to fulfill short-term liabilities. It is one of the company's liquidity indicators that characterizes its ability to withstand transient changes in the market and business environment, such as delays in payments by customers, fluctuations in sales, unforeseen expenses or claims for immediate payment of debts.

As a rule, it is believed that this coefficient should not be less than 1, however, the recommended values can vary significantly depending on the industry, country and other conditions.

Calculation formula:

$$CR = \frac{\text{Current assets}}{\text{current liabilities}}.$$

The satisfactory financial state of the enterprise usually meets the values of this coefficient that exceeds 1.6-2.0.

– *quick liquidity ratio (coefficient of litmus test, Quick Ratio, Acid Test Ratio, QR)* - a financial coefficient equal to the ratio of highly liquid current assets and current liabilities of the company.

It is calculated according to the company's balance sheet as the quotient of dividing the amount of funds, short-term investments and debts by current liabilities. It characterizes the company's ability to meet its current liabilities using the most liquid (convertible into money) assets.

Calculation formula:

$$QR = \frac{\text{Funds} + \text{short - term investments} + \text{debts}}{\text{current liabilities}},$$

– *absolute liquidity ratio* - shows how much short-term debt can be covered by funds and its equivalents in the form of marketable securities and deposits, i.e. almost completely liquid assets.

It is calculated by the formula:

$$AL = \frac{\text{Funds} + \text{short - term financial investments}}{\text{current liabilities}}.$$

Recommended values: 0.2 - 0.5

II. *Turnover ratios (used to assess the effectiveness of operating activities and policies in the field of prices, sales and purchase):*

– *asset turnover ratio (turnover ratio)* - a financial indicator calculated as the ratio of the company's turnover to the average annual total assets.

Calculation formula:

$$AT = \frac{\text{Sales volume}}{\text{average total assets}},$$

where *sales volume* – company annual turnover;

*Average total assets* - the average annual value of total assets (usually defined as the amount at the beginning and at the end of the year divided in half).

– *equity turnover ratio* - an indicator characterizing the rate of equity use and reflects the effectiveness of enterprise resource management.

The indicator of the equity turnover is used to assess various aspects of the enterprise functioning:

Commercial - the effectiveness of the sales system;

Financial - dependence on borrowed funds of the enterprise;

Economic - the intensity of equity use.

The formula for calculating the equity ratio:

$$K_{eqt} = \frac{\text{sales revenue}}{\text{equity}}.$$

High values of the equity turnover ratio are achieved due to an increase in sales revenue, and as a rule, this is due to the large share of profit obtained through the use of borrowed funds. As a result, in the long-term perspective, this can undermine the financial stability and independence of the enterprise from external sources of financing. Low values of the ratio reflect the fact that the equity of the enterprise is not being used effectively enough.

– *Inventory Turnover* — a financial indicator calculated as the ratio of the cost of goods sold to the average annual value of stocks.

Calculation formula:

$$IT = \frac{\text{cost of goods sold}}{\text{average stock}},$$

where *the cost of goods sold* – annual production costs;

*Average stocks* - the average annual value of stocks (usually defined as the amount at the beginning and at the end of the year, divided in half, although a more detailed study of their changes during the year is possible).

The higher the stock turnover of a company, the more efficient is production and the less is the need for working capital for its organization.

This indicator is calculated, as well as the average number of days during which stocks are at the warehouse. In this case, the formula will be following:

$$IT = \frac{\text{average stock}}{\text{cost of goods sold}} 365.$$

Using the data on the profit and losses not for the year, but for a different period, the cost of sales should be adjusted accordingly.

– *receivables turnover ratio* - a financial indicator calculated as the ratio of the company's turnover to the average annual amount of receivables.

Calculation formula:

$$RT = \frac{\text{sales volume}}{\text{average receivables}},$$

where *sales volume* – company annual turnover;

*Average accounts receivable* - the average annual value of accounts receivable (usually defined as the amount at the beginning and at the end of the year, divided in half, although a more detailed study of its changes during the year is possible).

RT shows how efficiently the company organized the collection of payments for its products. A decrease in this indicator may signal an increase in the number of insolvent customers and other sales problems, but may also be associated with the company's transition to a softer customer relationship policy aimed at expanding market share. The lower the turnover of receivables, the higher will be the company's need for working capital to expand sales.

Another variant of this indicator is widely spread, expressing the turnover of receivables as the average number of days required to collect debts. This variant of the indicator is called the *Collection Period (CP)* and is calculated using the following formula:

$$CP = \frac{\text{average accounts receivables}}{\text{sales volume}} 365.$$

Using the data on the profit and losses not for the year, but for a different period, the value of sales should be adjusted accordingly.

– *payables turnover ratio* - a financial indicator calculated as the ratio of the cost of goods sold to the average annual amount of accounts payable.

Calculation formula:

$$PT = \frac{\text{cost of goods sold}}{\text{average accounts payable}},$$

where *the cost of goods sold* – annual production costs.

*Average accounts payable* - the average annual value of accounts payable (usually defined as the amount at the beginning and at the end of the year divided in half, although a more detailed study of its changes during the year is possible).

The higher this indicator, the faster the company settles with its suppliers. A decrease in turnover can mean both problems with paying bills and a more efficient organization of relationships with suppliers, which provides a more profitable, deferred payment schedule and uses payables as a source of cheap financial resources.

Another variant of this indicator is widely spread, expressing the turnover of accounts payable as the average number of days during which the company pays its debts. This variant of the indicator is called Accounts payable collection period and is calculated using the following formula:

$$APCP = \frac{\text{average accounts payable}}{\text{cost of goods sold}} \cdot 365.$$

Using the data on the profit and losses not for the year, but for a different period, the cost of sales should be adjusted accordingly.

III. *Profitability indicators (used to assess the current profitability of the enterprise participating in the project):*

– *return on sales ratio (Net Profit Margin, ROS)* - the ratio of the company's net profit to its turnover.

Calculation formula:

$$ROS = \frac{\text{net profit}}{\text{sales volume}} \cdot 100\%.$$

All data necessary for the calculation of this indicator is taken from the profit and losses statement. The profitability ratio of sales reflects the company's income for each ruble earned and can be useful both for the correct interpretation of turnover data and for economic forecasts in the context of a limited market volume that constrains sales growth. Also, sales profitability is an important indicator for comparing the effectiveness of business organization in companies operating in the same industry.

– *return on Assets (ROA)* - the ratio of the company's net profit excluding interest on loans to its total assets.

The rate of return on assets characterizes the ability of the company's management to effectively use its assets for profit. In addition, this ratio reflects the average return received on all sources of capital (own and borrowed).

Calculation formula:

$$ROA = \frac{\text{net profit} + \text{interests}(1 - \text{tax rate})}{\text{average total assets}} \cdot 100\% ,$$

where *interests* – interest on loans (from profit and loss statement);

*Tax rate* - marginal income tax rate, i.e. the rate at which the additional income of the company will be taxed;

*Average total assets* - the average annual value of the total assets of the company (defined as the sum of assets at the beginning and at the end of the year divided in half).

Interest on loans is added to the net profit of the company in order to exclude their accounting from the ratio. This allows to make its calculation independent from the method of financing activities and facilitates the comparison of profitability indicators of different companies. Since the accrued interest in the profit and loss statement not only reduces the value of net profit, but also ensures a reduction in tax payments (by an amount equal to Interest \* Tax rate), for their correct accounting, the amount of interest must be adjusted using this formula, for this there is used the coefficient ( 1- Tax rate).

In practice, other ROA calculation variants are sometimes used. Sometimes the profit before taxes and interest (EBIT) is used as a numerator in the coefficient. With another, fairly common variant of the calculation, instead of profit without interest, there is used pure net profit. It should be noticed that in the last case, companies financed by borrowed capital will look less profitable than companies financed by equity, although their real effectiveness may be the same.

In all cases, at calculating this coefficient, it is assumed to use data from annual income statements. If quarterly or other statements are used in the calculation, the coefficient should be multiplied by the number of reporting periods in a year.

The specified list can be supplemented at the request of individual project participants or financial institutions, as well as in connection with the introduction by state bodies of new or changes to existing criteria for initiating bankruptcy proceedings.

The values of the relevant indicators should be analyzed in dynamics and compared with the indicators of similar enterprises. Each project participant, as well as lending banks and lessors, can have their own idea of the limit values of these indicators, indicating the unfavorable financial situation of the company. However, in any case, these limit values substantially depend on the production technology and the price structure of the products and resources consumed. Therefore, to use the prevailing at the time of calculating idea of the marginal levels of financial indicators to assess the financial situation of an enterprise over a long period of implementation of an investment project is not always advisable.

## **13.5 The analysis of investment efficiency**

### **13.5.1 Determination of the main indicators of the investment project**

There are several methods for assessing the attractiveness of an investment project (IP), in which various indicators are used to select a project. Each method is based on an important principle, according to which as a result of the project

implementation the investor must make a profit. Moreover, various financial indicators characterize the project from different sides in accordance with the interests of various interested parties associated with the investor.

The main criteria used in the analysis of investment activity (integral indicators) include [1-3, 9, 33]:

– *net present value (NPV) or net current value (NCV)* - cumulative discounted effect for the billing period:

$$NPV = -K + \sum_{t=1}^{t=T} \frac{R(t) - C(t)}{\prod_{s=1}^{s=t} (1 + E_s)}; \quad (13.1)$$

where  $K$  – initial investment;

$R(t)$  – money inflow in  $t$  year;

$C(t)$  – money outflow in  $t$  year;

$T$  – the duration of the life cycle;

$E_s$  – discount rate.

In order to determine  $NPV$ , it is necessary to select the discount rate and, based on its value, to find the appropriate discount factors for the analyzed billing period.

$NPV$  shows whether an investment has achieved the desired level of return over its economic life span:

$NPV > 0$ : accept IP, there is the possibility of additional income in excess of the normative profit

$NPV < 0$ : IP is unprofitable, projected cash receipts do not provide minimum regulatory return and investment recovery

$NPV = 0$ : IP will bring neither profit nor loss

– *internal rate of return (IRR)*, which is calculated as the root of the equation

$$-K + \sum_{t=1}^{t=T} \frac{R(t) - C(t)}{(1 + IRR)^t} = 0; \quad (13.2)$$

In IP, starting with investment costs and having a positive value of net money inflows,  $IRR$  is a positive number  $E_B$  if:

– at a discount rate of  $E_s = E_B$   $NPV = 0$ ,

– the number is singular.

In a more general case, the  $IRR$  is such a positive number  $E_B$  that, at a discount rate of  $E_s = E_B$   $NPV = 0$ , for all large values of  $E_s$  –  $NPV$  is negative, for all smaller values of  $E_s$  –  $NPV$  is positive. If at least one of these conditions is not met, then it is considered that the  $IRR$  does not exist.

IP is as follows:

– you must first carry out the cost of funds (allow the outflow of funds) and only then you can count on cash receipts (money inflows);

– cash receipts are cumulative in nature, and their sign changes only once (i.e., at first they can be negative, but, after becoming positive, will remain so throughout the entire calculation period).

For such IPs, the statement is true that the higher the discount rate ( $E_s$ ), the lower the value of the integral effect ( $NPV$ ).

$IRR$  is defined as the discount rate ( $E_s$ ) at which  $NPV = 0$ , i.e. IP does not ensure the growth of the value of the company, but does not lead to its decline.  $IRR$  is sometimes called a *verification discount*, since it allows to find the boundary value of the discount rate ( $E_g$ ), dividing the boundary investment into acceptable and unprofitable. For this, the  $IRR$  is compared with the discount rate ( $E_s$ ) accepted for the project:

- if  $IRR > E_s$ : the project is acceptable (i.e.  $NPV > 0$ );
- if  $IRR < E_s$ : the project is not acceptable (i.e.  $NPV < 0$ );
- if  $IRR = E_s$ : any decision can be made ( $NPV = 0$ ).

– *profitability index (PI) or discounted investment return index (DII)* - the ratio of the sum of discounted cash flow elements from operating activities to the absolute value of the discounted sum of cash flow elements from the investment activity:

$$PI = \frac{\sum_{t=1}^{t=T} \frac{R(t) - C(t)}{\prod_{s=1}^{s=t} (1 + E_s)}}{K}; \quad (13.3)$$

Calculating  $PI$ , either all capital investments for the billing period, including investments for the replacement of retiring fixed assets, or only initial capital investments made before the company is put into operation, can be taken into account.

– *payback period (PP)* - the minimum period of time after which the  $NPV$  becomes and remains non-negative:

$$PP = t_{n-1} + \frac{K - \sum_{t=1}^{t=T} R(t) - C(t)}{R(n) - C(n)}; \quad (13.4)$$

– *return on investment* - calculation of accounting return on investment (ROI) is based on income before interest and tax payments (EBIT) or income after tax, but before interest payments [EBIT (1 - H)]; as for the value of investments, in relation to which profitability is determined, it is found as the average between the recorded value of assets at the beginning  $c_n^1$  and end  $c_n^2$  of the period under review

$$ROI = \frac{EBIT (1-H)}{(C_n^1 + C_n^2) / 2}. \quad (13.5)$$

– the public effectiveness of the project ( $OP$ ) characterizes the socio-economic consequences of the project for society as a whole, i.e. it takes into account not only the immediate outcomes and costs of the project, but also the costs and outcomes “external” to the project in related sectors of the economy, economic, social and other non-economic effects, as a rule, a qualitative assessment is given ( $0 < OP < 1$ ).

All calculations of efficiency indicators (integral indicators) are performed with discounted cash flows representing cash inflows, or money receipts, and cash outflows, or money payments, during the project implementation.

The  $NPV$  indicator reflects a direct increase in the company's capital, therefore it is the most significant for the shareholders of the company. The criterion for the adoption of the project for implementation is the positive value of  $NPV$  at a predetermined discount rate.

Internal rate of return ( $IRR$ ) is independent from the discount rate. To calculate it, you can use computer tools or manually calculate it by the selection method. According to the rule of internal rate of return, an investment project should be accepted if the opportunity cost of raising capital is less than the internal rate of return, that is,  $IRR > \alpha$ . If the opportunity cost is equal to the internal rate of return, then this is a project with zero  $NPV$ .

Comparing several projects, projects with large  $IRR$  values are more preferable.

$PI$  profitability index must be greater than 1, which means the attractiveness of investing in a project. Considering several projects, you should choose the one that has the highest profitability index for implementation.

The payback period  $PP$  in assessing the effectiveness of an investment project acts as a limiting condition. In some cases, there is used a project classification system in which the payback period indicates how quickly each project should be implemented.

The public performance  $OP$  for an investment project is a qualitative assessment, the value of which ranges from zero to one. The most significant investment projects for society should have a value of  $OR$  close to unity. Determining this, the possible results from the implementation of the project should be taken into account: revenue from sales of products, environmental and social results, financial results (taxes, depreciation, profit), etc.

### 13.5.2 The algorithm for investment projects selection

Let there are many investment projects:

$$A = \{A[i]\}, i = 1, 2, \dots, n$$

for each of which the above efficiency indicators are calculated, forming the set

$$P = \{P[j]\}, j = 1, 2, \dots, m$$

The values of these  $m$  indicators for the considered  $n$  investment projects are presented in the form of a matrix:

$$P = \begin{pmatrix} P_{11} & P_{12} & \dots & P_{1m} \\ P_{21} & P_{22} & \dots & P_{2m} \\ \dots & \dots & \dots & \dots \\ P_{n1} & P_{n2} & \dots & P_{nm} \end{pmatrix} \quad (13.6)$$

In relation to the indicators considered in subsection 13.5.1, the matrix can be represented in the form of table 13.1.

Table 13.1 – Data table

$A_i \backslash P_j$	$NPV$	$IRR$	$PI$	$PP$	$ROI$	$OP$
$A_1$	$P_{11}$	$P_{12}$	$P_{13}$	$P_{14}$	$P_{15}$	$P_{16}$
$A_2$	$P_{21}$	$P_{22}$	$P_{23}$	$P_{24}$	$P_{25}$	$P_{26}$
...	...	...	...	...	...	...
$A_n$	$P_{n1}$	$P_{n2}$	$P_{n3}$	$P_{n4}$	$P_{n5}$	$P_{n6}$

In order to select from a variety of projects  $A$  one or more projects with the best indicators, you can use the classical methods of voting theory (Pareto, Borda, Condorcet). Considering that the number of selected projects is not always predetermined, the most acceptable in this case, according to the authors, is the Borda method [28]. The essence of the Borda method is to determine the voting results as the number of points scored by each of the candidates. So, in elections from  $n$  candidates, each voter ranks all candidates strictly in descending order of preference. For the first place in preference, the candidate is awarded by  $n$  points, for the second -  $n-1$  points, etc. (for the last place - 1 point), all points scored by candidates are summed up. Accordingly, the candidate with the highest total score is considered to be the winner of the election.

As applied to the problem of investment projects selection, where strict ranking of the values of all indicators is not always possible, the algorithm of this method can be created in the following form [49]:

*Step 1.* Let  $j = 1$ .

*Step 2.* Ranking the elements of the  $j$ -th column of the matrix (13.6) in descending order of the values of its elements.

*Step 3.* Assigning the highest rank  $r=n$  to the first element of the ranked sequence, the rank  $r=n-1$  to the next element of the sequence, etc., before the completion of the assignment procedure according to the characteristics  $r = 1$  or  $r = k$ , where  $k$  is the number of elements of the  $j$ -th column of the matrix of indicators identical with any other elements of this column

*Step 4.* Verifying the condition  $j \leq m$ . If yes, go to step 5, otherwise – to step 6.

*Step 5.* Calculating  $j = j+1$  and go to step 2.

*Step 6.* For each project  $A[i]$ ,  $i = 1, 2, \dots, n$  determination of its rank  $r[i]$  as the total value for all columns.

*Step 7.* Display the results.

The ranks of investment projects obtained in this way are the basis for selection of a subset of  $B \in A$ , including projects suitable for investment from the initial set  $A$ .

### **13.6 Development of methods and algorithms for computer analysis of the distribution of funds by various priorities**

In modern conditions of significant volatility in global financial markets and a high level of uncertainty and risks in the implementation of investment projects, there is an increasing need to develop methods for assessing and managing the value and risks of investment projects. One of the main tasks of the investment programs of the state development bank is the distribution of funds, exactly investment, according to various priorities and criteria determined by the decisions of the Government of the Republic of Kazakhstan.

#### 13.6.1 Optimization models to achieve a given IP efficiency

One of the cardinal ways to reduce risk arising from the portfolio theory of the Nobel laureate G. Markowitz is to solve the optimization problem to achieve a given investment portfolio (IP) efficiency at the lowest risk value:

$$\begin{aligned} V_p &= \sum_{j=1}^n x_j^2 \sigma_j^2 \rightarrow \min ; \\ \sum_{j=1}^n B_j x_j &= B_p ; \\ \sum_{j=1}^n x_j &= 1. \end{aligned} \tag{13.7}$$

The Markowitz model is based on the fact that the profitability indicators of various securities are interrelated: with the growth of the profitability of some securities, there is a simultaneous increase in other securities, the third remain unchanged, and on the fourth, on the contrary, the profitability decreases. This type of dependence is not deterministic, i.e. uniquely defined, but stochastic, and is called *a correlation*.

The next step to reduce the risk of the securities portfolio (SP) is to include risk-free government securities in the portfolio investment object. The essence of this approach, obtained by another Nobel laureate Tobin, is as follows: "If it is possible to choose not only between a given risk portfolio and risk-free securities, but also to choose a risk portfolio structure, then only one such structure will be optimal, regardless of the investor's inclination to risk".

The mathematical model of Tobin's portfolio optimization is:

$$\begin{aligned}
V_p &= \sum_{j=1}^n x_j^2 \sigma_j^2 \rightarrow \min ; \\
x_0 B_0 + \sum_{j=1}^N B_j x_j &= B_p ; \\
x_0 + \sum_{j=1}^N x_j &= 1,
\end{aligned} \tag{13.8}$$

where  $x_j$  – percentage of  $j$ -th security in portfolio;  
 $B_j$  – profitability of the  $j$ -th security;  
 $B_0$  – profitability of the risk-free part of the portfolio;  
 $x_0$  – percentage invested in the risk-free part of the portfolio;  
 $B_p$  – set value of portfolio profitability indicator.

### 13.6.2 Optimization models based on diversification characteristics

Let suppose a bank (or another investor) has  $S$  million dollars to invest in securities of companies from  $K$  different sectors of the economy. Each company  $i$  from sector  $k$  undertakes to pay dividends (or interest) on these securities for a period of time, after which it undertakes to repay the loan amount. Suppose company 1 agrees to pay \$ 0.025 dollars per quarter for 60 quarters for every dollar of the loan. Another company promises to pay an increasing percentage for 30 quarters. The third company promises to pay the full amount with accrued interest after 40 quarters, etc.

Since companies may have difficulties in future, the Bank considers these payments as random variables, which may be more or less than promised. The task of the bank is to place investment resources in such a way as to maximize its expected return on the portfolio, but the following restrictions are taken into account:

- 1) in order to diversify, the bank's management requires that the debts of a company from one sector do not exceed 20% of the portfolio;
- 2) placing an investment portfolio, the bank must consider and limit the risk of losing money.

Formally, the task of the bank may look as follows:

minimization of expected variation in the amount of future payments:

$$\min_{x_i^k \geq 0} \text{var} \left[ \sum_{k=1}^K \sum_{i=1}^{n^k} (x_i^k \sum_{t=0}^T \beta^t \gamma_{it}^k) \right] \tag{13.9}$$

taking into account the expected net present value (NPV):

$$E_0 \left[ \sum_{k=1}^K \sum_{i=1}^{n^k} (x_i^k \sum_{t=0}^T \beta^t \gamma_{it}^k) \right] \geq B \tag{13.10}$$

and fulfillment of diversification restrictions:

$$\sum_{i=1}^{n^k} x_i^k \leq A_k S \text{ for all sectors } k = \overline{1, K}, \quad (13.11)$$

$$\sum_{k=1}^K \sum_{i=1}^{n^k} x_i^k = S \quad (13.12)$$

The following notations are used in the task:

- 1)  $x_i^k \geq 0$  – investments in company  $i$  from the sector  $k$ , where  $i = 1, 2, \dots, n^k$ ;
- 2)  $\beta \in (0, 1)$  – discount factor that the bank uses to measure payments from subsequent quarters;
- 3) index  $T$  – maximum number of quarters during which at least some companies must repay loans (maximum loan term);
- 4)  $K$  – number of different sectors. And the number  $n^k$  – the number of companies from sector  $k$  that the bank is considering to include in its portfolio;
- 5)  $A_k$  – maximum percentage of investments in companies from the  $k$  sector;
- 6)  $\gamma_{it}^k$  – payment of company  $i$  from sector  $k$  in a quarter  $t$ . The Bank accepts values as random parameters of its task (market interest rates);
- 7)  $B$  – expected return on bank portfolio.

13.6.3 Mathematical models of the distribution of funds while maximizing expected income

Different types of securities can be attributed to different groups of investment risk.

*1st group - low risk.* This group may include ordinary bonds, current bank accounts, bank deposit certificates, etc. However, such “safe” investments, from the point of view of risk, give a small income.

*2nd group - medium risk.* The second group may include ordinary shares. Income from such securities is higher, but it is subject to significant fluctuations, which increases the risk.

*3rd group - high risk.* This group may include various “speculative shares”. The rate of such securities tends to fluctuate strongly, which increases the risk, but the expected return on them can be quite high. The policy of the bank is that the bank allocates from the total amount of investments certain shares of funds for investments in securities of various groups.

Therefore, the boards of many investment firms consider it necessary to invest a certain part of investments in low-risk securities. This restriction is written as follows:

$$\sum_{j \in J_1} x_j \geq r_1 S,$$

where  $J_1$  – many indices of securities of the 1st group;  $r_1$  – minimum share of investments in securities of the 1st group.

On the other hand, most investment firms limit the size of investments in ordinary and especially in “speculative” shares, since the income from them is subject to significant fluctuations. Such restrictions are written as follows:

$$\sum_{j \in J_2} x_j \geq r_2 S,$$

$$\sum_{j \in J_3} x_j \geq r_3 S,$$

where  $J_2, J_3$  – respectively, the set of indices of securities of the 2nd and 3rd groups;  $r_2, r_3$  – respectively, the maximum share of investments in securities of the 2nd and 3rd groups.

Thus, the mathematical model for the formation of the optimal portfolio will take the form:

$$B_p = \sum_{j=1}^N x_j B_j \rightarrow \max,$$

$$\sqrt{\sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij}} \leq V_p,$$

$$\sum_{j=1}^n x_j \leq S,$$

$$\sum_{j \in J_1} x_j \geq r_1 S,$$

$$\sum_{j \in J_2} x_j \geq r_2 S,$$

$$\sum_{j \in J_3} x_j \geq r_3 S,$$

$$x_j \geq 0,$$

where  $x_j$  – number of investments in the  $j$ -th security in the portfolio;

$B_j$  – profitability of the  $j$ -th security;

$B_p$  – set value of portfolio profitability indicator;

$V_p$  – set value of portfolio risk indicator;

$S$  – the volume of investments in the portfolio;

$\sigma_{ij}$  – covariance of securities  $i$  and  $j$ ;

$J_1, J_2, J_3$  – set of securities indices of the 1st, 2nd and 3rd risk groups;  
 $r_1$  – minimum share of investments in securities of the 1st group;  
 $r_2, r_3$  – maximum shares of investments in securities of the 2nd and 3rd risk groups.

#### 13.6.4 Payment distribution analysis

The formulated task (13.9) - (13.12) is formally a quadratic programming task, however, in practice, solving it, a number of problems arise associated with the uncertainty and randomness of its individual parameters.

Agreeing on the amount of payments, the tasks between the investor and the company usually focus not only on the individual profitability component of a particular company, but also on the total profitability component of the entire market  $R^M$  and the average profitability of companies of the corresponding sector  $r^k$ . Therefore, the interest rate  $\gamma_{it}^k$  can be decomposed as follows [50]:

$$\gamma_{it}^k = b_{ik}^M R_t^M + c_i^k r_t^k + \varepsilon_{it}^k, \quad (13.13)$$

where  $b_{ik}^M$  and  $c_i^k$  determine how sensitive the incomes of company  $i$  from sector  $k$  are to changes in the entire market or in a separate sector (factor loadings), and  $\varepsilon_{it}^k$  – is the individual profitability component of a company  $i$  from sector  $k$ . Coefficients  $b_{ik}^M$  and  $c_i^k$  are usually found using simple linear regression.

Let assume that  $\varepsilon_{it}^k$  is independent from the individual profitability components of other companies, so for  $i \neq j$ :

$$\text{cov}\left(\left(\sum_{t=0}^T \beta^t \varepsilon_{it}^k\right), \left(\sum_{t=0}^T \beta^t \varepsilon_{jt}^m\right)\right) = 0.$$

It is also assumed that  $r_t^k$  is independent from the sectoral profitability components of other sectors, therefore, for  $k \neq m$ :

$$\text{cov}\left(\left(\sum_{t=0}^T \beta^t r_t^k\right), \left(\sum_{t=0}^T \beta^t r_t^m\right)\right) = 0.$$

Based on the above assumptions, the expression for the expected value of  $NPV$  can be reduced to the following form:

$$E_0[NPV] = E_0\left[\sum_{k=1}^K \sum_{i=1}^{n^k} \left(x_i^k \sum_{t=0}^T \beta^t \gamma_{it}^k\right)\right] = E_0\left[\sum_{k=1}^K \sum_{i=1}^{n^k} \left(x_i^k \sum_{t=0}^T \beta^t (b_{ik}^M R_t^M + c_i^k r_t^k + \varepsilon_{it}^k)\right)\right] =$$

$$\begin{aligned}
&= \left\{ E_0 \left[ \sum_{k=1}^K \sum_{i=1}^{n^k} \left( x_i^k b_{ik}^M \sum_{t=0}^T \beta^t (R_t^M) \right) \right] + E_0 \left[ \sum_{k=1}^K \sum_{i=1}^{n^k} \left( x_i^k c_i^k \sum_{t=0}^T \beta^t (r_t^k) \right) \right] + E_0 \left[ \sum_{k=1}^K \sum_{i=1}^{n^k} \left( x_i^k \sum_{t=0}^T \beta^t (\varepsilon_{it}^k) \right) \right] \right\} = \\
&= \left\{ E_0 \left[ \sum_{t=0}^T \beta^t (R_t^M) \sum_{k=1}^K \sum_{i=1}^{n^k} (x_i^k b_{ik}^M) \right] + E_0 \left[ \sum_{k=1}^K \sum_{t=0}^T \beta^t (r_t^k) \sum_{i=1}^{n^k} (x_i^k c_i^k) \right] + E_0 \left[ \sum_{k=1}^K \sum_{i=1}^{n^k} x_i^k \sum_{t=0}^T \beta^t (\varepsilon_{it}^k) \right] \right\} = \\
&= E_0 \left[ Y^M \sum_{t=0}^T \beta^t (R_t^M) \right] + E_0 \left[ \sum_{k=1}^K Y^k \sum_{t=0}^T \beta^t (r_t^k) \right] + E_0 \left[ \sum_{k=1}^K \sum_{i=1}^{n^k} x_i^k \sum_{t=0}^T \beta^t (\varepsilon_{it}^k) \right],
\end{aligned}$$

where:

$$Y^M = \sum_{k=1}^K \sum_{i=1}^{n^k} (x_i^k b_{ik}^M), \quad Y^k = \sum_{i=1}^{n^k} (x_i^k c_i^k).$$

Similarly, we transform the expression for variation *NPV*:

$$\text{var} = \left[ \sum_{k=1}^K \sum_{i=1}^{n^k} \left( x_i^k \sum_{t=0}^T \beta^t \gamma_{it}^k \right) \right] = \text{var} \left\{ \left[ Y^M \sum_{t=0}^T \beta^t (R_t^M) \right] + \left[ \sum_{k=1}^K Y^k \sum_{t=0}^T \beta^t (r_t^k) \right] + \left[ \sum_{k=1}^K \sum_{i=1}^{n^k} x_i^k \sum_{t=0}^T \beta^t (\varepsilon_{it}^k) \right] \right\}$$

which, by the assumption of independence of distributions, is equal to

$$\begin{aligned}
&= (Y^M)^2 \text{var} \left[ \sum_{t=0}^T \beta^t (R_t^M) \right] + \sum_{k=1}^K (Y^k)^2 \text{var} \left[ \sum_{t=0}^T \beta^t (r_t^k) \right] + \sum_{k=1}^K \sum_{i=1}^{n^k} (x_i^k)^2 \text{var} \left[ \sum_{t=0}^T \beta^t (\varepsilon_{it}^k) \right] = \\
&= (Y^M)^2 \sigma_M^2 + \sum_{k=1}^K (Y^k)^2 \sigma_k^2 + \sum_{k=1}^K \sum_{i=1}^{n^k} (x_i^k)^2 \sigma_{ik}^2,
\end{aligned}$$

where

$$\sigma_M^2 = \text{var} \left[ \sum_{t=0}^T \beta^t (R_t^M) \right], \quad \sigma_k^2 = \text{var} \left[ \sum_{t=0}^T \beta^t (r_t^k) \right], \quad \sigma_{ik}^2 = \text{var} \left[ \sum_{t=0}^T \beta^t (\varepsilon_{it}^k) \right].$$

With these transformations, problem (13.9) - (13.12) takes the form:

$$F = (Y^M)^2 \sigma_M^2 + \sum_{k=1}^K (Y^k)^2 \sigma_k^2 + \sum_{k=1}^K \sum_{i=1}^{n^k} (x_i^k)^2 \sigma_{ik}^2 \rightarrow \min, \quad (13.14)$$

$$\begin{aligned}
&= E_0 \left[ Y^M \sum_{t=0}^T \beta^t (R_t^M) \right] + E_0 \left[ \sum_{k=1}^K Y^k \sum_{t=0}^T \beta^t (r_t^k) \right] + E_0 \left[ \sum_{k=1}^K \sum_{i=1}^{n^k} x_i^k \sum_{t=0}^T \beta^t (\varepsilon_{it}^k) \right] \geq B, \\
&(13.15)
\end{aligned}$$

$$\sum_{i=1}^{n^k} x_i^k \leq A_k S, \quad k = \overline{1, K}, \quad (13.16)$$

$$\sum_{k=1}^K \sum_{i=1}^{n^k} x_i^k = S. \quad (13.17)$$

The resulting task (13.14) - (13.17) is a task with “perturbed” parameters, since interest rates  $\varepsilon_{it}^k$  play the role of small perturbations here. As it is known, the solution of such optimization problems is associated with the instability of the obtained solutions [54]. To solve the task (13.9) - (13.12) with “perturbed” parameters, there is proposed a method for expanding the set of feasible solutions [57] based on the formation of an extended investment portfolio optimization task:

$$\min_{x_i^k \geq 0} \text{var} \left[ \sum_{k=1}^K \sum_{i=1}^{n^k} \left( x_i^k \sum_{t=0}^T \beta^t \gamma_{it}^k \right) \right], \quad (13.18)$$

$$\sum_{k=1}^K \sum_{i=1}^{n^k} x_i^k = S \quad (13.19)$$

and establishing the relationship between the solutions of the initial “perturbed” (13.9) - (13.12) and extended (13.18) - (13.19) tasks. This method allows to find exact and stable solutions to tasks with small perturbations and, in particular, the task (13.9) - (13.12) for specific values of its parameters. However, forming an investment portfolio, it is also necessary to take into account the uncertainty of a number of parameters of this task. For example, the impact of changes in the parameters  $A_k$ , which determine the strategy of diversification, on the effectiveness of the investment portfolio. To achieve this goal, we consider several approaches: simulation, statistical and analytical.

### 13.6.5 Diversification coefficient modeling

The general structure of simulation modeling for analyzing the parameters of the task of forming the optimal investment portfolio should ensure the implementation of the following main stages [61].

1. Solution of the task (13.9) - (13.12).
2. Modeling of possible changes in diversification coefficients.
3. Determining the range of variation of diversification coefficients leading to an improvement in the optimal solution to the task (13.9) - (13.12).

In accordance with the given structure of the simulation model, the following search algorithm  $\Delta A_k^*$  can be proposed.

*Step 1.* Accept  $j = 0$ .

*Step 2.* Solution of the task (13.10)-(13.13) and finding portfolio profitability

*B.*

*Step 3.* Calculating the left-hand sides of constraints (13.11), i.e. expressions

$$\sum_{i=1}^{n^k} x_i^k, \quad k = \overline{1, k}, \quad (13.20)$$

corresponding to the optimal solution to the task (13.9) - (13.12). in case of the repeated solution of this problem, go to step 7.

*Step 4.* Determination of the effective constraints of the task (13.9)-(13.12).

*Step 5.* Modeling possible changes in diversification coefficients  $\Delta A_k$  for effective constraints.

*Step 6.* Calculating new effective constraint values:

$$A_k^{j+1} = A_k^j + \Delta A_k.$$

$j = j + 1$ . Go to step 2.

*Step 7.* Comparison of the values of the objective functions and profitability of the portfolio of two successively solved tasks (13.9) - (13.12). in case of improvement of the value of the objective function and profitability, go to step 4.

*Step 8.* Determining the area of expedient change of task parameters

$$\Delta A_k^* = A_k^{j-1} - A_k^0,$$

where  $A_k^0$  – the values of the left-hand sides of constraints (13.11) corresponding to the optimal solution to the task (13.9)-(13.12).

*Step 9.* End of the algorithm.

### 13.6.6 Prediction of changes in diversification parameters

Predicting changes in diversification parameters for a certain period of time in future is one of the important tasks of forming optimal investment portfolios of the bank. The most common type of forecast is based on the methods of statistical analysis [51]. Here, in turn, one of the main methods is the least squares method. As a predictor, linear (13.21), cyclic (13.22), or linear-cyclic (13.23) functions are used

$$A'_{k,t} = a + bt; \quad (13.21)$$

$$A'_{k,t} = a + u \cos \frac{2\pi}{N} t + \vartheta \sin \frac{2\pi}{N} t; \quad (13.22)$$

$$A'_{k,t} = a + bt + u \cos \frac{2\pi}{N} t + \vartheta \sin \frac{2\pi}{N} t. \quad (13.23)$$

The construction of the predictive function is shown by the example of a linear function (13.21). For this, on the basis of the least squares method, it is

necessary to minimize the sum of the squared differences between the actual diversification coefficient  $A_{k,t}$  at the moment of time  $t$  and the predictive function:

$$E = \sum_{t=1}^n (A_{k,t} - a - bt)^2. \quad (13.24)$$

Further, equating the partial derivatives to zero, we obtain the equations:

$$\frac{\partial E}{\partial a} = \sum (A_{k,t} - a - bt) = 0; \quad (13.25)$$

$$\frac{\partial E}{\partial b} = \sum t(A_{k,t} - a - bt) = 0. \quad (13.26)$$

The determinant of the system of equations (13.21), (13.25) and (13.26) has the form:

$$\begin{vmatrix} A_k & 1 & t \\ \sum A_k & n & \sum t \\ \sum A_k t & \sum t & \sum t^2 \end{vmatrix}.$$

Then, solving this system of equations, we obtain

$$a = \frac{\sum A_k \sum t^2 - \sum t \sum A_k t}{n \sum t^2 - (\sum t)^2};$$

$$b = \frac{n \sum A_k t - \sum A_k \sum t}{n \sum t^2 - (\sum t)^2}.$$

Introducing the assumption that  $t$  takes sequentially integer values from 1 to  $n$ , we have

$$\sum t = \frac{n(n+1)}{2};$$

$$\sum t^2 = \frac{n(n+1)(2n+1)}{6}.$$

Consequently,

$$a = \frac{2(2n+1)\sum A_k - 6\sum A_k t}{n(n-1)};$$

$$b = \frac{6[2\sum A_k t - (n+1)\sum A_k]}{n(n^2-1)}.$$

Thus, there is described the procedure for predicting changes in diversification parameters based on the method of statistical analysis for a certain period of time in future.

Let create an algorithm for predicting changes in diversification coefficients  $A_k$ . [27]

*Step 1.* Initial data set.

*Step 2.* Cycle organization:  $i = 1$ .

*Step 3.* Calculating:

$$a = \frac{2(2n+1)\sum A_k - 6\sum A_k t}{n(n-1)}; \quad b = \frac{6[2\sum A_k t - (n+1)\sum A_k]}{n(n^2-1)}.$$

*Step 4.* Calculating diversification coefficient  $A'_{k,t}$ :  $A'_{k,t} = a + bt$

*Step 5.* Cycle extension:  $i = i + 1$ .

*Step 6.* Verifying the condition:  $i \leq n$ . If the condition is met, go to step 3.

*Step 7.* Display the result  $A'_{k,t}$ .

*Step 8.* End.

### 13.6.7 Analytical method for determining the $\Delta A_k^*$ [61]

Since the analysis of the model of the task (13.9) - (13.12) for sensitivity is performed relative to the diversification parameter  $A_k$ , then all other parameters of the task are assumed to be constant. Consequently, the values of the objective function  $Z^{paccu}$  and the values of the right-hand sides  $A_k^{paccu}$  of constraints (13.11) corresponding to the solution of the extended task (13.18) - (13.19) will be constant. This, in turn, leads to a nonlinear dependence of the value of the objective function (taken with the opposite sign) on the vector of the right-hand sides of the constraint (13.11) or, taking into account that  $S$  is a constant, on the vector  $A_k$  (Fig. 13.5) and to a nonlinear dependence of the portfolio profitability value on the vector of the right-hand sides of the constraint (13.11) or taking into account that  $S$  is a constant, on the vector  $A_k$  (Fig. 13.6).

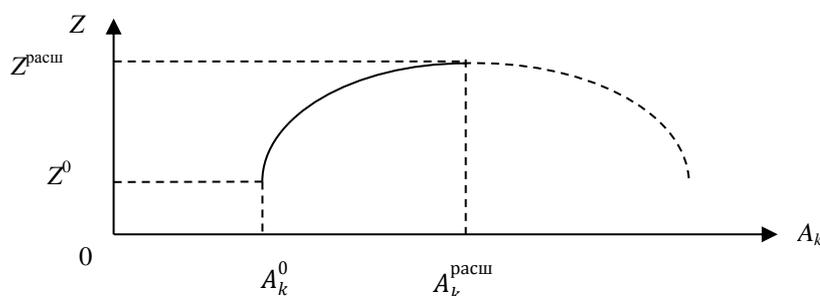


Figure 13.5 – The range of changes in the risk of the bank portfolio

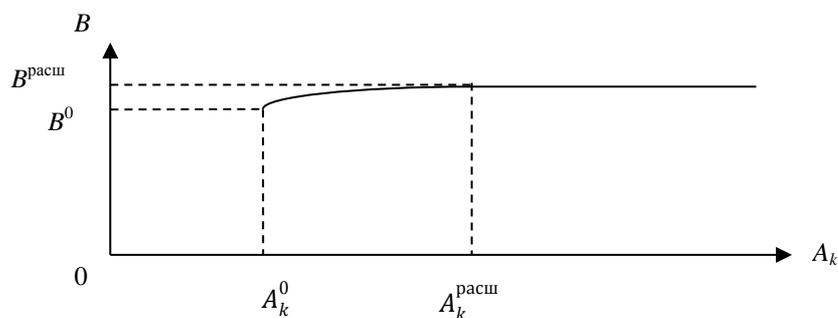


Figure 13.6 – The range of changes in the profitability of the bank portfolio

Here  $A_k^0$  – the value of the right-hand side of the constraint (13.11) corresponding to the optimal solution to the task (13.9) - (13.12), and  $A_k^{расши}$  – the value of the right-hand side of the constraint (13.11) corresponding to the solution of the extended task (13.18) - (13.19).

From the figure 13.5 it can be seen that the range of changes of the parameter  $A_k$ , in which it is possible to improve the found solution, is the interval  $[A_k^0, A_k^{расши}]$ , i.e. choosing a new value  $A_k$ , the condition  $A_k = A_k^0 + \Delta A_k \leq A_k^{расши}$  should not be violated.

As experience in solving optimization problems shows, in practice, not all restrictions of the form (13.11) are effective. For clarity, we consider the case when two constraints are effective. Let the parameters  $A_l^0$  and  $A_e^0$  correspond to these restrictions. In addition, we suppose that one of these parameters relates to a more important sector of the economy and is the main one (for example,  $A_l$ ). Change the values of these parameters:  $A_l^0 \rightarrow A_l^1$ ,  $A_e^0 \rightarrow A_e^1$  (Figure 13.7).

Point C corresponds to such a change in the diversification parameters on the graphs. The value of the objective function was improved:  $Z = Z^1$ .

Let then  $A_l^0 \rightarrow A_l^1$ , a  $A_e^0 \rightarrow A_e^1$ , although  $A_e^2 < A_e^1$ .

In the  $Z^2$  case (point D on the figure 13.7)  $Z = Z^2 (< Z^1)$ , and the constraints corresponding to the main parameter  $A_l^1$  becomes ineffective.

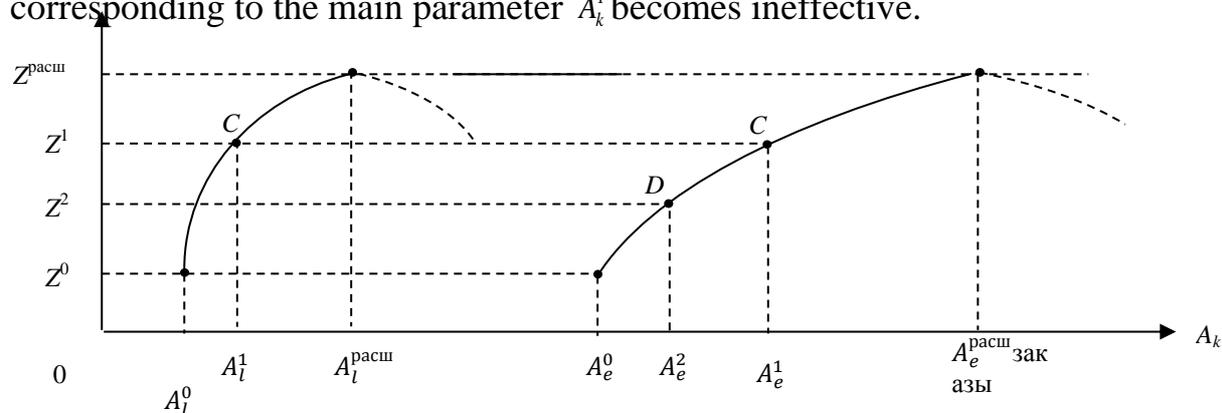


Figure 13.7 – Mutual influence of diversification parameters

The aforesaid proves the obviousness of the following statement: “The greatest effect from a change in resources will be achieved if in the new solution the constraints that were effective before were also effective.”

It is easy to verify that the formula  $A_e = A_e^{pacu} - \Delta A_k$  for a given change of the main parameter ( $\Delta A_k$ ) allows to find new values of the remaining parameters (effective constraints) at which the effect of changing the diversification parameters will be maximum.

The search algorithm for optimal  $\Delta A_k^*$  in bank portfolio optimization tasks, including simulation and analytical approaches, consists of the following steps.

*Step 1.* Initial data set.

*Step 2.* The choice of a method for determining the area of changes of non-stationary parameters: simulation or analytical. Choosing an analytical method, go to step 17.

*Step 3.* Accept  $j = 0$ .

*Step 4.* Solution the task (13.9)-(13.12) and finding portfolio profitability  $B$ .

*Step 5.* Calculating according to the formulas (13.20).

*Step 6.* In case of re-solving the task (13.9)-(13.12), i.e.  $j \neq 0$ , go to step 15.

*Step 7.* Determining the effective constrains of the task (13.9)-(13.12).

*Step 8.* In the case of discreteness of possible changes in non-stationary parameters for effective constraints, go to a step 10.

*Step 9.* Calculating  $\Delta A_k = F^{-1}(U)$ . Go to step 14.

*Step 10.* Accept  $m = 1$ .

*Step 11.* Verifying the condition  $U > \sum_{j=1}^m p_j$ . If no – go to step 17.

*Step 12.* Accept  $m = m + 1$ . Go to step 11.

*Step 13.* Accept  $\Delta A_k = \Delta A_{k,m}$ .

*Step 14.* Calculating of new effective constraint values:  $A_k^{j+1} = A_k^j + \Delta A_k$ ,  $j = j + 1$ . Go to step 4.

*Step 15.* Comparison of the values of the objective functions and portfolio profitability of two successively solved tasks (1-4). In case of improvement in the value of the objective function and profitability, go to a step 7.

*Step 16.* Determining the area of expedient change of diversification coefficients  $\Delta A_k^* = A_k^{j-1} - A_k^0$ . Go to step 19.

*Step 17.* Solution of the extended task (13.18)-(13.19) of bank IP optimization.

*Step 18.* Determining effective constraints of the task and the area of changes in the parameters by the formula:  $\Delta A_k^* = \Delta A_k^{pacu} - A_k^0$ .

*Step 19.* End of the algorithm.

### **13.7 Development of methods or algorithms for unformalized risks expert assessments**

One of the important risks of an investment project is the possibility of a lack

of its public effectiveness or even a contradiction to the public interests of the results of investment activities. This risk refers to unformalized indicators of investment projects and for their assessment it is necessary to use the knowledge of experts. The public effectiveness of the project (OP) characterizes the socio-economic consequences of the project for society as a whole, i.e. it takes into account not only the immediate results and costs of the project, but also the “external” social and other non-economic effects of the project.

Public effectiveness (OP) for an investment project is a quality assessment, the value of which ranges from zero to one. The most significant investment projects for society should have a value of  $OP$  close to unity. Determining this, the possible results from the implementation of the project should be taken into account: revenue from sales of products, environmental and social results, financial results (taxes, depreciation, profit), etc. There are three possible ways to determine the values of  $OP$ :

- the use of the Cherkmen – Akof expert assessment method [2];
- modeling possible values of  $OP$  as a multidimensional random variable with known distribution laws of these components (social and environmental results, financial results, revenue from sales of products, etc.);
- construction of a predictive function [51].

Determination of the values of public effectiveness by the method of expert assessments. Let there are many investment projects

$$A = \{A_i, i = 1, 2, \dots, n\}$$

The value of the social utility of each of them is denoted by  $OP_i, i = 1, 2, \dots, n$ .

In order to determine the specific values of  $OP_i$ , based on the Cherkmen-Akof utility measurement method, we create the following modified algorithm

Step 1. The numbering of projects is being arranged so that project  $A_1$  is the most, and the project  $A_n$  is the least preferred. consequently,  $OP_i > OP_{i+1}, i = 1, 2, \dots, n-1$ .

Step 2. Interval  $[0, 1]$  of possible values of  $OP$  is divided into  $n$  equal sub-intervals of length  $\Delta$ .

Step 3. On the basis of expert assessments, preference signs are put in the Cherkman-Akof table

Step 4. The preliminary estimates of the  $OP$  are calculated by the formula

$$OP_i = 1 + \Delta(1 - i), i = 1, 2, \dots, n$$

Step 5. Substituting the found preliminary values of  $OP_i$  into the inequalities of the Cherkman-Akof table in the reverse sequence correct the violated inequalities with a minimum step  $\Delta/2$ . The final obtained values  $OP_i, i = 1, 2, \dots, n$  are the values of the public efficiency of the projects under consideration  $A_i, i = 1, 2, \dots, n$ .

Table 13.1 - Cherkman-Akof table

$OP_1$ or $OP_2 + OP_3 + \dots + OP_n$	$OP_2$ or $OP_3 + OP_4 + \dots + OP_n$	.	$OP_{n+2}$ or $OP_{n-1} + OP_n$
$OP_1$ or $OP_2 + OP_3 + \dots + OP_{n-1}$	$OP_2$ or $OP_3 + OP_4 + \dots + OP_{n-1}$	.	END
$OP_1$ or $OP_2 + OP_3 + \dots + OP_{n-2}$	$OP_2$ or $OP_3 + OP_4 + \dots + OP_{n-2}$	.	
.	.	.	
.	.	.	
.	.	.	
$OP_1$ or $OP_2 + OP_3$	$OP_2$ or $OP_3 + OP_4$	.	
		.	

*Modeling the possible values of public efficiency.* If there are theoretical or obtained on the basis of statistical data patterns of changes of the mentioned above components of social efficiency, we can simulate their possible forecast values and thereby assess the estimated values of the public effectiveness of investment projects.

Let introduce the public effectiveness of the  $OP$  of a particular investment project in the form of a vector random variable  $OP = (P_1, P_2, \dots, P_m)$ , where  $P_i$  is the  $i$ -th component of public efficiency, for example,  $P_1$  - social results,  $P_2$  - environmental results, etc., and  $F_i(p)$  - distribution laws of values the  $i$ -th component.

In order to assess the possible values of public efficiency, it is necessary to simulate random variables  $P_i$ ,  $i = 1, 2, \dots, n$ , i.e. to find their implementation  $p_i$  according to given distribution laws. Under the conditions that the distribution laws are represented by simple analytical dependencies, the inverse function method described in Section 3 can be used. This method, despite a good theoretical justification, has a limited area of application because of the need to solve the integral equation.

In order to model the components  $P_i$ ,  $i = 1, 2, \dots, n$ , the distribution laws of which are represented by rather complicated analytical dependencies, or are given in the form of graphs or tables and the restriction  $p \in [a, b]$  is imposed, there can be applied the John von Neumann exclusion method (section 3 of this textbook), which has a more applied character.

After determining in this way possible implementations  $p$  of all the components  $P_i$ ,  $i = 1, 2, \dots, n$  experts assess the value of the  $OP$  of a particular project.

### Conclusion

The analytical and simulation approaches to investment decision-making processes presented in this chapter and the developed methods and algorithms for computer modeling of these processes are quite universal and relevant not only for banks, but also for any other organizations involved in investment activities. They are especially applicable in the activities of pension funds, national funds, as well as in the work of bank supervision structures, where it is very important to take into

account various aspects of risk and uncertainty of the main parameters at optimizing the portfolio.

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**COMPUTER MODELLING.  
METHODS AND APPLICATIONS**

Tutorial

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