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ALMATY UNIVERSITY OF POWER ENGINEERING AND TELECOMMUNICATIONS

The department of
Higher mathematics

MATHEMATICS 1

Summary of lectures

for students of specialties
5B071700 «Heat power engineering»,
5B071800 «Electrical power engineering»,
5B071900 «Radio engineering, electronics and telecommunications»

Almaty 2014

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The present summary of lectures contains 12 lectures on the basic sections in the discipline Mathematics 1: «The elements of linear algebra, analytic geometry and complex numbers», «Differential calculus of one variable functions», «Integral calculus of one variable function» and corresponds to the curriculum of the specialties 5B071700 «Heat power engineering», 5B071800 «Electrical power engineering», 5B071900 «Radio engineering, electronics and telecommunications».

The theoretical material is illustrated by examples and figures.

The summary of lectures is intended for the first-year students. It may also be useful for the self-study.

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Reviewer: candidate of sciences in philology, V.S. Kozlov

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Preface

The present summary of lectures contains 12 lectures on the basic sections, traditionally studied in the discipline Mathematics 1: «The elements of linear algebra, analytic geometry and complex numbers», «Differential calculus of one variable functions», «Integral calculus of one variable function» and corresponds to the curriculum of the specialties 5B071900, 5B071800, 5B071700.

The content of sections is interconnected with each other. The theoretical material is illustrated by examples and pictures.

The summary of lectures is intended for the first-year students. It may also be useful for the self-study.

1 The elements of linear algebra, analytic geometry and complex numbers.

1.1 Lecture 1. Matrices. Determinants. Systems of linear equations

Content of the lecture: Matrices. Inverse matrix. 2-nd and 3-rd order determinants, its properties. Algebraic complements and minors. n-th order determinants. Systems of two and three linear equations with 2 or 3 unknowns. Matrix notation of the system of linear equations. Solution of the system of linear equations by Cramer's rule and by matrix method.

Aims of the lecture: study the basic concepts of linear algebra.

1. Matrices

A matrix is a rectangular table consisting of numbers or functions. There are two types of matrices: numerical and functional.

A matrix $A_{m\times n}$ of size $m\times n$ is written in parentheses:

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

where m is the number of rows, n is the number of columns, a_{ij} is an element of the matrix, i, j – indexes, $i=\overline{1,m}$ (from 1 to m), $j=\overline{1,n}$ (from 1 to n).

Types of matrices:

1) A matrix of size $1\times n$ is called a row matrix. A matrix of size $m\times 1$ is called a column matrix. Column matrix and row matrix are called vectors.

2) Matrices of the same size $A_{m\times n}$ and $B_{m\times n}$ are called equal, if

$$a_{ij} = b_{ij} \qquad \forall\ i=\overline{1,m},\quad j=\overline{1,n}.$$

3) A null-matrix is a matrix in which all elements are zero:

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};\quad 0 = \begin{pmatrix} 0 & 0 & \dots & 0 \end{pmatrix}.$$

4) If $m = n$, then the matrix is square.

A square matrix D is called diagonal if its diagonal elements may contain non-zero elements, but all nondiagonal elements are zero:

$$D=\begin{pmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_{nn} \end{pmatrix}.$$

The identity matrix is a diagonal matrix with ones along the diagonal:

$$E=\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 1\end{pmatrix}.$$

A square matrix $A_{n\times n}$ is called symmetric, if

$$a_{ij} = a_{ji} \qquad \forall\ i=\overline{1,n},\quad j=\overline{1,n}.$$

Operations on matrices

1. Addition.

Matrices of the same size can be added (elementwise):

$$A+B=C=\begin{pmatrix}c_{11} & c_{12} & \dots & c_{1n}\\ c_{21} & c_{22} & \dots & c_{2n}\\ \dots & \dots & \dots & \dots\\ c_{m1} & c_{m2} & \dots & c_{mn}\end{pmatrix}, \qquad c_{ij}=a_{ij}+b_{ij} \quad i=\overline{1,m};\quad j=\overline{1,n}.$$

2. Multiplication of the matrix by any number α.

Product of the matrix $A_{m\times n} = (a_{ij})$ by any number α (or number α by the matrix $A_{m\times n}$) is the matrix $B_{m\times n} = A\alpha = \alpha A = (b_{ij})$:

$$b_{ij} = \alpha a_{ij}, \qquad i=\overline{1,m},\quad j=\overline{1,n}.$$

$(-1)\,A$ is the opposite matrix to the matrix A.

3. Subtraction.

Subtraction is defined as $A - B = A + (-1)B.$

4. Multiplication of matrices.

If the sizes of the matrices are respectively

$$A_{m\times n} = (a_{ij}) \quad \text{and} \quad B_{n\times k} = (b_{ij})$$

(ie the number of columns of the first matrix equals the number of rows of the second matrix) then these matrices can be multiplied as follows:

$$C = A\cdot B = (c_{ij}) = C_{m\times k}, \qquad c_{ij} = \sum_{s=1}^{n} a_{is}b_{sj}$$

(in this case, we say that matrix $A_{m\times n} = (a_{ij})$ is compatible with the matrix $B_{n\times k} = (b_{ij})$).

5. Degree of matrix.

For the degrees of square matrix $A_{n\times n} = (a_{ij})$ the following notations are used:

$$A^0 = E,\quad A^1 = A,\quad A^2 = A\cdot A,\quad A^3 = A\cdot A\cdot A,\ \dots$$

6. Inverse matrices.

Suppose that A is a square matrix: $A_{n\times n} = (a_{ij})$, $i=\overline{1,m}$; $j=\overline{1,n}$.
The inverse matrix for A is the matrix A^{-1} for which $A\cdot A^{-1} = E$ or $A^{-1}\cdot A = E$.
The existence of the inverse matrix depends on its determinant.

Determinant of the matrix A.

Determinant of the matrix is denoted by det $A = |A| = \Delta$.

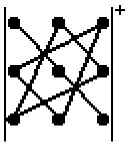
1) The second-order determinant is the number:

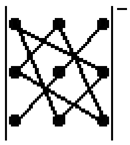
$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

The second order determinant is the product of the elements of the main diagonal minus the product of the elements of the secondary diagonal.

2) The third-order determinant is the number:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

The triangle rule: the third-order determinant is the sum of six terms; terms with a plus sign are obtained by multiplication of three elements of the determinant taken by the scheme  ,

terms with a negative sign – by the scheme  .

Matrix A is invertible (nonsingular) $\Leftrightarrow$ $\det A \neq 0$.

The minor M_{ij} of an element a_{ij} is the determinant of order lower by one consisting of the elements that remain after the deletion of the i-th row and j-th column which intersect in a_{ij}.

The algebraic complement of the element is calculated by formula

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

Calculation of the inverse matrix.

Consider a square nonsingular matrix $A_{n\times n} = (a_{ij})$, (ie $|A| \neq 0$). Then the inverse matrix A^{-1} is calculated by the formula

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}.$$

2. Systems of linear equations.

Consider a system of n linear equations with n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = h_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = h_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = h_n \end{cases}, \qquad (1)$$

a_{ij}, $h_i \in R$, $i = \overline{1,n}$.

a_{ij} - coefficients of the equations, h_i - free terms.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$ - the basic matrix of the system,

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & h_1 \\ a_{21} & a_{22} & \dots & a_{2n} & h_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & h_n \end{bmatrix}$$ - the extended matrix of the system.

If we introduce the following notation: $X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$, $H = \begin{bmatrix} h_1 \\ h_2 \\ . \\ . \\ . \\ h_n \end{bmatrix}$,

the system (1) can be written in matrix form: $AX = H$.

Vector $C = (c_1;\ c_2;\ ...;\ c_n)$ is called the vector-solution of the system if $AC = H$.

Solution of nonsingular linear systems.

Consider the system (1). Suppose that $\Delta = \det A \neq 0$.

1) Cramer's rule.

$$\boxed{x_i = \frac{\Delta_i}{\Delta},} \quad i = \overline{1, n}.$$

The auxiliary determinants $\Delta_i \ \left(i = \overline{1, n}\right)$ are obtained from the determinant of the basic matrix of the system by replacing the *i*-th column by column of free terms.

2) Matrix method for solving system of linear equations.

Since the system (1) is written as a matrix equation $AX = H$, then

$$A^{-1}AX = A^{-1} H.$$

Since $A^{-1}A = E$ is the identity matrix we have

$$X = A^{-1}H$$

1.2 Lecture 2. Vectors

Content of the lecture: Three-dimensional space R^3. Vectors. Linear operations over vectors. Scalar product, vector product, triple product in R^3 and its properties. Modulus of vector. Angle between two vectors.

Aims of the lecture: study the basic concepts of vectors theory.

1. Vectors.

A directed segment (or an ordered pair of points) is called a vector $\overrightarrow{AB}$.

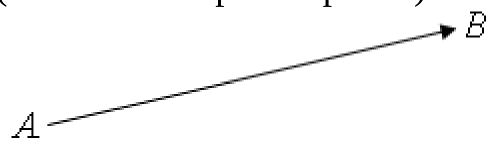


Figure 1

Vector with coinciding endpoints is called the null vector.

The distance between the head and tail of the vector is called the length (the module, the norm or the absolute value) of this vector.

It is denoted by $|\vec{a}|$ or $|\overrightarrow{AB}|$.

The vector of unit length is called the unit vector. It is denoted by $|\vec{e}| = 1$.

Vectors are collinear if they lie on a line or on parallel lines.

Vectors are coplanar if they lie in the same plane or in parallel planes.

Two vectors are equal if they are collinear and have the same direction and length.

We denote by φ the angle formed by two vectors $\vec{a}$ and $\vec{b}$ having a common head $(0 \le \varphi \le \pi)$.

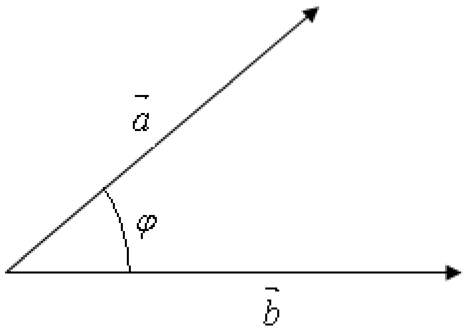


Figure 2

Depending on the angle φ we have the following definitions:

two vectors $\vec{a}$ and $\vec{b}$ are codirected if $\varphi = 0$;

two vectors $\vec{a}$ and $\vec{b}$ are oppositely directed if $\varphi = \pi$;

two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if $\varphi = \pi/2$.

Linear operations on vectors.

1) addition $\vec{a} + \vec{b}$;

2) multiplication by a scalar (number) $\vec{b} = \alpha\vec{a}$;

3) subtraction $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

Cartesian coordinate system in space.

Consider the following coordinate system: take mutually perpendicular unit vectors $\vec{i}, \vec{j}, \vec{k}$ emanating from a point O. This system is denoted by $(O; \vec{i}, \vec{j}, \vec{k})$ or $Oxyz$, where

O – the origin of coordinates,

Ox – x-axis,

Oy – y-axis,

Oz – z-axis,

Ox, Oy, Oz – the coordinate axes.

The radius vector of the point $M(\alpha_1, \alpha_2, \alpha_3)$: $\overrightarrow{OM} = \alpha_1\vec{i} + \alpha_2\vec{j} + \alpha_3\vec{k}$.

Coordinates of a vector $\overrightarrow{AB}$.

Two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are given in space.

Using the radius vectors of the given points, we have $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, therefore $\overrightarrow{AB}(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

Actions on vectors with given its coordinates.

Consider the vectors $\vec{a} = \alpha_1 \vec{i} + \alpha_2 \vec{j} + \alpha_3 \vec{k}$ and $\vec{b} = \beta_1 \vec{i} + \beta_2 \vec{j} + \beta_3 \vec{k}$.

1) Addition: $\vec{a} + \vec{b} = (\alpha_1 + \beta_1)\vec{i} + (\alpha_2 + \beta_2)\vec{j} + (\alpha_3 + \beta_3)\vec{k}$.

2) Subtraction: $\vec{a} - \vec{b} = (\alpha_1 - \beta_1)\vec{i} + (\alpha_2 - \beta_2)\vec{j} + (\alpha_3 - \beta_3)\vec{k}$.

3) Multiplication by a number: $\lambda\vec{a} = (\lambda\alpha_1)\vec{i} + (\lambda\alpha_2)\vec{j} + (\lambda\alpha_3)\vec{k}$.

Collinearity condition of two vectors: $\vec{b} = \lambda\vec{a} \Leftrightarrow \dfrac{\alpha_1}{\beta_1} = \dfrac{\alpha_2}{\beta_2} = \dfrac{\alpha_3}{\beta_3}$.

2. Inner product, vector product, triple product in R^3 and its properties.

1) Inner product.

The inner product of two vectors $\vec{a}$ and $\vec{b}$ is the product of the absolute values of these vectors and the cosine of the angle between them:

$$\boxed{\left(\vec{a},\vec{b}\right) = |\vec{a}| \cdot \left|\vec{b}\right| \cdot \cos\varphi,} \qquad 0 \le \varphi \le \pi$$

Since

$$\left|\vec{b}\right| \cdot \cos\varphi = np_{\vec{a}}\vec{b},$$

$$|\vec{a}| \cdot \cos\varphi = np_{\vec{b}}\vec{a},$$

then $\left(\vec{a},\vec{b}\right) = |\vec{a}| \cdot np_{\vec{a}}\vec{b} = \left|\vec{b}\right| \cdot np_{\vec{b}}\vec{a}$

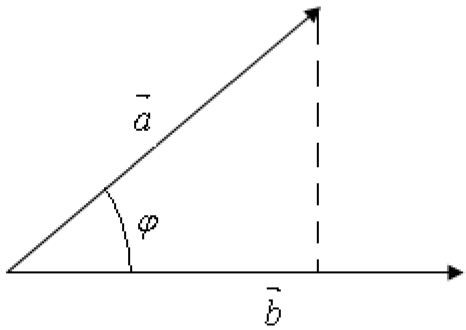


Figure 3

Properties of the inner product:

1) $\left(\vec{a},\vec{b}\right) = \left(\vec{b},\vec{a}\right)$ (commutativity);

2) $\left(\lambda \cdot \vec{a},\vec{b}\right) = \lambda \cdot \left(\vec{a},\vec{b}\right)$, $(\lambda \in R)$;

3) $\left(\vec{a}+\vec{b},\vec{c}\right) = \left(\vec{a},\vec{c}\right) + \left(\vec{b},\vec{c}\right)$ (distributivity);

4) $\left(\vec{a},\vec{a}\right) = \vec{a}^2 = |\vec{a}|^2$.

Theorem 1. (orthogonality of two vectors)

$$\vec{a} \perp \vec{b} \quad \Leftrightarrow \quad \left(\vec{a},\vec{b}\right) = 0 .$$

The inner product in coordinate form.

Consider two vectors: $\vec{a} = \alpha_1\vec{i} + \alpha_2\vec{j} + \alpha_3\vec{k}$ and $\vec{b} = \beta_1\vec{i} + \beta_2\vec{j} + \beta_3\vec{k}$.
Then the following formulas are true

$$\left(\vec{a},\vec{b}\right) = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3;$$

$$(\vec{a},\vec{a}) = |\vec{a}|^2 = \alpha_1^{\ 2} + \alpha_2^{\ 2} + \alpha_3^{\ 2};$$

$$|\vec{a}| = \sqrt{\alpha_1^{\ 2} + \alpha_2^{\ 2} + \alpha_3^{\ 2}};$$

$$\cos\varphi = \frac{(\vec{a},\vec{b})}{|\vec{a}|\cdot|\vec{b}|}.$$

The orthogonality condition of two vectors in coordinate form.

$$\vec{a} \perp \vec{b} \quad \Leftrightarrow \quad \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0$$

2) Vector product.

The vector product of two vectors $\vec{a}$ and $\vec{b}$ is a vector $\vec{c} = [\vec{a},\vec{b}]$ satisfying the following conditions:

1) $\vec{c} \perp \vec{a}, \quad \vec{c} \perp \vec{b};$

2) three vectors $\vec{a},\vec{b},\vec{c}$ constitute a right triple of vectors (that is, looking from the tail of vector $\vec{c}$, we see that the shorter rotation from $\vec{a}$ to $\vec{b}$ is carried out anticlockwise);

3) $|\vec{c}| = |\vec{a}|\cdot|\vec{b}|\cdot\sin\varphi$.

Properties of the vector product:

1) $[\vec{a},\vec{b}] = -[\vec{b},\vec{a}]$ (anticommutativity);

2) $[\lambda\vec{a},\vec{b}] = \lambda[\vec{a},\vec{b}], \quad (\lambda \in R);$

3) $[\vec{a}+\vec{b},\vec{c}] = [\vec{a},\vec{c}] + [\vec{b},\vec{c}]$ **(**distributivity**).**

Theorem 2.

If $a \neq 0, \ b \neq 0,$ then $[\vec{a},\vec{b}] = 0 \Leftrightarrow \vec{b} = \lambda\vec{a}$.

The vector product in coordinate form.

Consider two vectors: $\vec{a} = \alpha_1\vec{i} + \alpha_2\vec{j} + \alpha_3\vec{k}$ and $\vec{b} = \beta_1\vec{i} + \beta_2\vec{j} + \beta_3\vec{k}$.

Then the following formula is true

$$[\vec{a},\vec{b}]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\end{vmatrix}=\begin{vmatrix}\alpha_2 & \alpha_3\\ \beta_2 & \beta_3\end{vmatrix}\vec{i}-\begin{vmatrix}\alpha_1 & \alpha_3\\ \beta_1 & \beta_3\end{vmatrix}\vec{j}+\begin{vmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{vmatrix}\vec{k}.$$

3) Triple product.

The triple product of three vectors is the inner product of the third vector by the vector product of the first two vectors; it is denoted by

$$(\vec{a},\vec{b},\vec{c})=\left([\vec{a},\vec{b}],\vec{c}\right).$$

Theorem 3.

Consider three no coplanar vectors: $\vec{a}=\overrightarrow{AB},\ \vec{b}=\overrightarrow{AC},\ \vec{c}=\overrightarrow{AD}$.

1) if $\vec{a},\vec{b},\vec{c}$ is a right triple of vectors, then $(\vec{a},\vec{b},\vec{c})=V_{a,b,c}$;

2) if $\vec{a},\vec{b},\vec{c}$ is a left triple of vectors, then $(\vec{a},\vec{b},\vec{c})=-V_{a,b,c}$;

That is $|(\vec{a},\vec{b},\vec{c})|=V_{a,b,c}$,

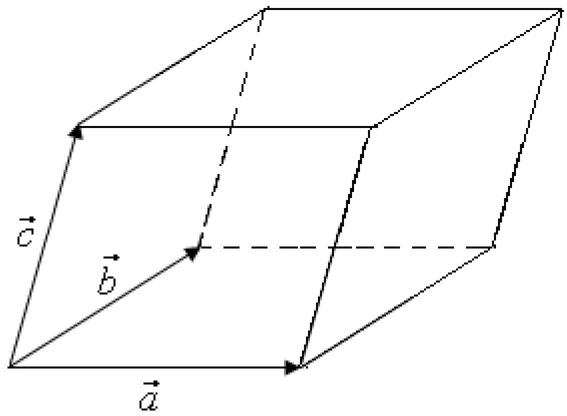


Figure 4

where $V_{a,b,c}$ is the volume of a parallelepiped spanned by three vectors $\vec{a},\vec{b},\vec{c}$.

Corollary.

It is easy to derive an expression for the volume of a pyramid spanned by three vectors $\vec{a},\vec{b},\vec{c}$:

$$V_{pyr}=\frac{1}{6}\left|(\vec{a},\vec{b},\vec{c})\right|.$$

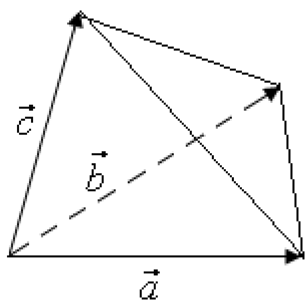


Figure 5

Theorem 4._(coplanarity condition of three vectors)

Consider three vectors: $\vec{a}\neq 0,\vec{b}\neq 0,\vec{c}\neq 0$.

Three vectors $\vec{a},\vec{b},\vec{c}$ are coplanar $\Leftrightarrow \left(\vec{a},\vec{b},\vec{c}\right)=0$.

Properties of the triple product:

1) $(\vec{a},\vec{b},\vec{c})=(\vec{b},\vec{c},\vec{a})=(\vec{c},\vec{a},\vec{b})$;

2) $(\lambda\vec{a},\vec{b},\vec{c})=(\vec{a},\lambda\vec{b},\vec{c})=(\vec{a},\vec{b},\lambda\vec{c})=\lambda(\vec{a},\vec{b},\vec{c})$.

The triple product in coordinate form.

Consider three vectors: $\vec{a}=\alpha_1\vec{i}+\alpha_2\vec{j}+\alpha_3\vec{k}$, $\vec{b}=\beta_1\vec{i}+\beta_2\vec{j}+\beta_3\vec{k}$ and $\vec{c}=\gamma_1\vec{i}+\gamma_2\vec{j}+\gamma_3\vec{k}$.

Then the following formula is true

$$(\vec{a},\vec{b},\vec{c})=\begin{vmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{vmatrix}.$$

1.3 Lecture 3. Planes in space

Content of the lecture: Different types of equations of plane in R^3. The distance from a point to a plane.

Aims of the lecture: study the basic concepts and equations of plane in space R^3.

1. The equation of a plane passing through a given point perpendicular to a given vector.

Suppose given a vector $\vec{n}(A,B,C)$ perpendicular to a plane and a point $M_0(x_0, y_0, z_0)$ in this plane. This vector is called a normal vector.

Then the equation of this plane has the form:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0.$$

Since $|\vec{n}|\neq 0$, then $|A|+|B|+|C|\neq 0$.

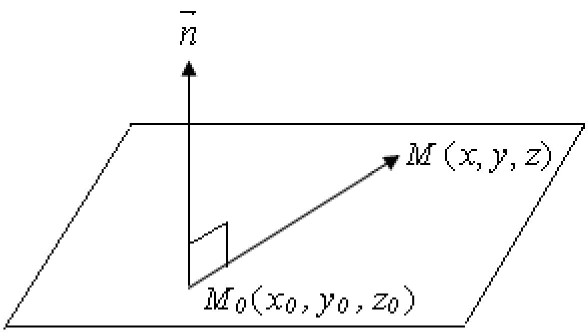


Figure 6

2. The general equation of a plane.

$$Ax+By+Cz+D=0,$$

where $D=-(Ax_0+By_0+Cz_0)$, $|A|+|B|+|C|\neq 0$.

3. The equation of a plane passing through three given points.

Suppose given three points $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$, $M_3(x_3, y_3, z_3)$. We take an arbitrary point $M(x, y, z)$ in the plane.

The characteristic feature of a plane is that if a point M belongs to the plane, then three vectors

$\overrightarrow{M_1M}=(x-x_1; y-y_1; z-z_1)$,

$$\overrightarrow{M_1M_2} = (x_2 - x_1; y_2 - y_1; z_2 - z_1),$$
$$\overrightarrow{M_1M_3} = (x_3 - x_1; y_3 - y_1; z_3 - z_1)$$

are coplanar.

Therefore, the triple product of these vectors must be zero.

So we obtain an equation of the plane passing through three points:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

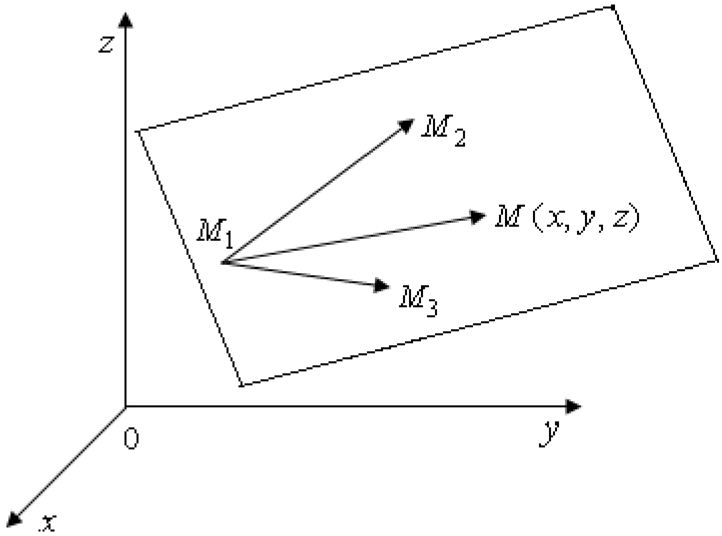


Figure 7

4. The equation of a plane in segments.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

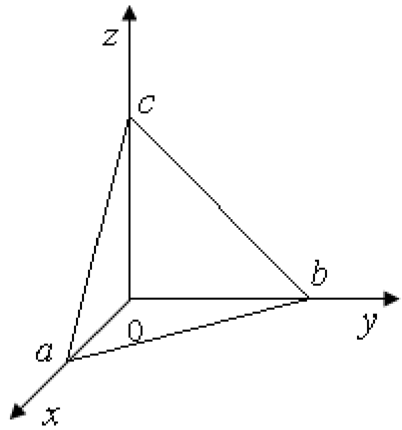


Figure 8

5. The angle between two planes.

Consider two planes given by the equations

$A_1 x + B_1 y + C_1 z + D_1 = 0,$

$A_2 x + B_2 y + C_2 z + D_2 = 0,$

which have normal vectors $\overrightarrow{n_1}(A_1, B_1, C_1)$ $\overrightarrow{n_2}(A_2, B_2, C_2)$. Using inner product, we find the cosine of the angle:

$$cos\varphi = \pm \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

The condition of parallelism of two planes: $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$.

The condition of perpendicularity of two planes: $A_1A_2 + B_1B_2 + C_1C_2 = 0$.

6. The distance from a point to a plane.

The distance from a point $M_1(x_1, y_1, z_1)$ to a plane:

$$Ax + By + Cz + D = 0,$$

can be calculated by the formula:

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

1.4 Lecture 4. Straight lines in the plane and in space

Content of the lecture: Straight line equations in R^2 and in R^3 .

Aims of the lecture: study the basic concepts and different types of straight line equations in plane R^2 and in space R^3.

Straight lines in the plane.

1) The equation of a straight line passing through a given point perpendicular to a given vector.

Suppose that a straight line passes through the point $M_0(x_0, y_0)$ perpendicular to the vector $\vec{n}(A, B)$, then the equation of this straight line has the form:

$A(x - x_0) + B(y - y_0) = 0.$

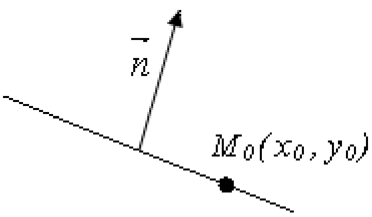


Figure 9

2) The general equation of a straight line:

$$Ax + By + C = 0,$$

where $C = -Ax_0 - By_0$, $|A| + |B| \neq 0$.

3) The canonical equation of a straight line:

$$\frac{(x - x_0)}{m} = \frac{(y - y_0)}{n},$$

where $\vec{s}(m, n)$ is the direction vector.

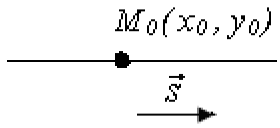


Figure 10

If the direction vector $\vec{s}(m, n)$ and y-axis are not parallel, ie $m \neq 0$, then $\frac{n}{m} = tg\alpha = k$,

where α is an angle between the straight line and the x-axis (measured anticlockwise from the positive direction of the x-axis to this line).
$tg\ \alpha$ is called the slope of the straight line.

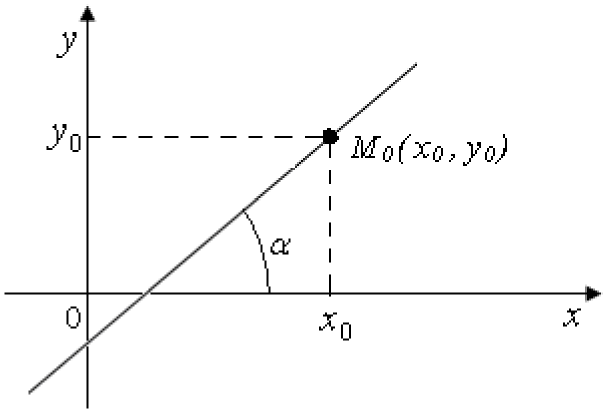


Figure 11

4) The equation of a straight line with a slope:

$$y - y_0 = k(x - x_0) \,.$$

5) The equation of a straight line passing through two given points.

Suppose that a straight line passes through two points M_1 (x_1, y_1) and M_2 (x_2, y_2), then the equation of this straight line has the form: $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$

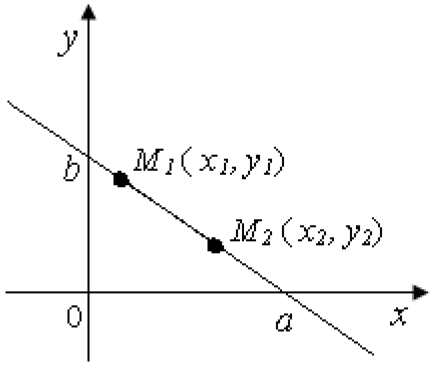


Figure 12

6) The equation of a straight line in segments:

$$\frac{x}{a} + \frac{y}{b} = 1$$

7) The angle between two straight lines.

Suppose that two straight lines are given:

$$l_1: \quad \frac{x - x_1}{m_1} = \frac{y - y_1}{n_1}; \qquad l_2: \quad \frac{x - x_2}{m_2} = \frac{y - y_2}{n_2}.$$

Then the formula for the cosine of the angle between these lines has the form:

$$\cos\varphi = \left| \frac{m_1 m_2 + n_1 n_2}{\sqrt{m_1^2 + n_1^2}\sqrt{m_2^2 + n_2^2}} \right|.$$

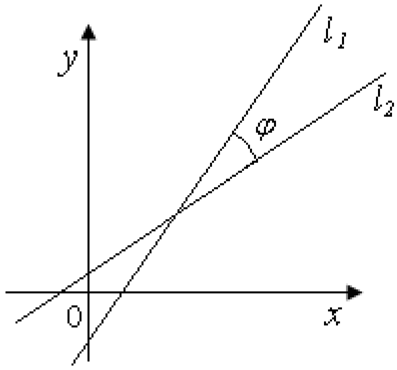


Figure 13

The condition of parallelism of two straight lines: $\frac{m_1}{m_2} = \frac{n_1}{n_2}$.

The condition of perpendicularity of two straight lines: $m_1 m_2 + n_1 n_2 = 0$.

Straight lines in space.

1. The equation of a straight line passing through a given point parallel to a given vector.

Suppose given a direction vector $\vec{s}(m, n, p)$ and a point $M_0(x_0, y_0, z_0)$ in space. Then the equation of a straight line passing through a given point parallel to a given vector has two forms:

the parametric equation: $$\begin{cases} x = x_0 + mt, \\ y = y_0 + nt, \\ z = z_0 + pt, \end{cases} \quad t \in R;$$

the canonical equations: $$\frac{(x - x_0)}{m} = \frac{(y - y_0)}{n} = \frac{(z - z_0)}{p}.$$

2. The general equation of a straight line in space.

Since a straight line in space is represented as the intersection of two planes, the general equation of a straight line in space has the form of a system:

$$\begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$$

where the first and the second equations are the equations of the corresponding planes.

3. The angle between two straight lines.

Suppose that two straight lines are given:

$$l_1: \quad \frac{x - x_1}{m_1} = \frac{y - y_1}{n_1} = \frac{z - z_1}{p_1}; \qquad l_2: \quad \frac{x - x_2}{m_2} = \frac{y - y_2}{n_2} = \frac{z - z_2}{p_2}.$$

Then the formula for the cosine of the angle between these lines has the form:

$$cos\varphi = \pm \frac{m_1 m_2 + n_1 n_2 + p_1 p_2}{\sqrt{m_1^2 + n_1^2 + p_1^2} \cdot \sqrt{m_2^2 + n_2^2 + p_2^2}}.$$

The condition of parallelism of two straight lines:

$$\frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{p_1}{p_2}.$$

The condition of perpendicularity of two straight lines:

$$m_1 m_2 + n_1 n_2 + p_1 p_2 = 0 .$$

4. The angle between a straight line and a plane.

Suppose that plane in space is determined by its general equation

$$Ax + By + Cz + D = 0,$$

and a straight line is defined by its canonical equations

$$\frac{(x - x_0)}{m} = \frac{(y - y_0)}{n} = \frac{(z - z_0)}{p} .$$

Suppose that β is the angle between the normal vector $\vec{n}(A, B, C)$ and the direction vector $\vec{s}(m, n, p)$; φ is the angle between the straight line and the plane.

Then we have the following equality:

$$\cos \beta = \cos (\pi/2 \pm \varphi) = \sin \varphi .$$

Thus, the formula for the sine of the angle between the straight line and the plane has the form:

$$\sin \varphi = \left| \frac{Am + Bn + Cp}{\sqrt{A^2 + B^2 + C^2} \cdot \sqrt{m^2 + n^2 + p^2}} \right| .$$

The perpendicularity condition for a straight line and a plane:

$$\vec{n} // \vec{s} \Leftrightarrow \frac{A}{m} = \frac{B}{n} = \frac{C}{p} .$$

The parallelism condition for a straight line and a plane:

$$n \perp s \Leftrightarrow Am + Bn + Cp = 0.$$

5. The distance from a point to a straight line on a plane.

The distance from a point $M_1 (x_1, y_1)$ to a straight line:

$$Ax + By + C = 0,$$

can be calculated by the formula:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} .$$

1.5 Lecture 5. Second-order curves in the plane

Content of the lecture: Second-order curves. Canonical form of equations of ellipse, hyperbola and parabola. Geometrical properties of ellipse, hyperbola and parabola.

Aims of the lecture: study the basic concepts, equations and geometrical properties of ellipse, hyperbola and parabola.

Ellipse.

An ellipse is the locus of points in the plane for which the sum of distances to two fixed points is constant and equal $2a$. These points are called the foci and denoted by F_1, F_2. Let $M(x, y)$ be an arbitrary point.

$|F_1F_2| = 2c$,

$|MF_1| + |MF_2| = 2a, \quad a > c$.

Introduce the notation: $b = \sqrt{a^2 - c^2}$, $0 < b \le a$.

The canonical equation of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (0 < b \le a)$$

a – major semi-axis of the ellipse;
b – minor semi-axis of the ellipse.

Construction of an ellipse.

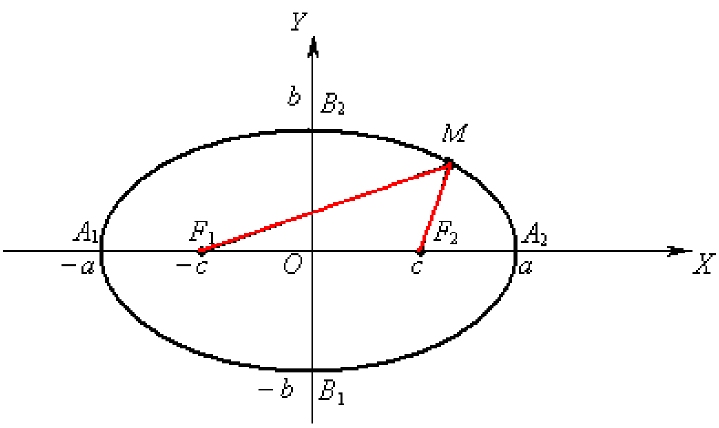


Figure 14

O – center of the ellipse,

$A_1(-a, 0)$, $A_2(a, 0)$,
$B_1(0, -b)$, $B_2(0, b)$
– the tops of the ellipse.

Segment $[A_1A_2]$ and its length $|A_1A_2| = 2a$ – the major axis of the ellipse.
Segment $[B_1B_2]$ and its length $2b$ – the minor axis of the ellipse.

In the considered case $(a > b)$ the foci $F_1(-c, 0)$, $F_2(c, 0)$ are on the ax is Ox, between A_1 and A_2.

If $a < b$, then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$ – the equation of an ellipse with foci on the axis Oy:

$F_1(0, -c)$, $F_2(0, c)$,

$$a=\sqrt{b^2-c^2}.$$

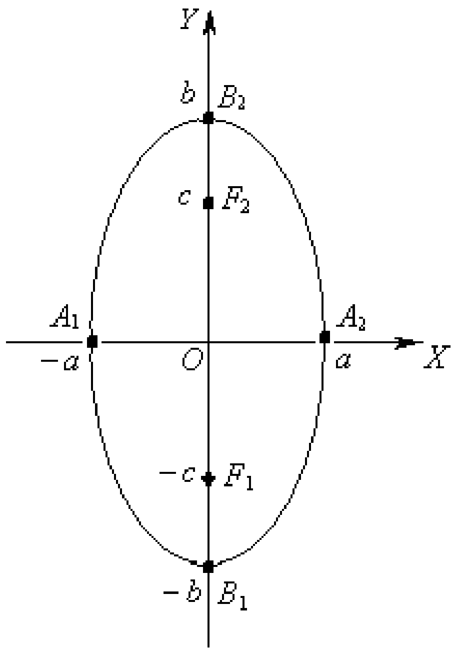


Figure 15

If $a = b$, then

$$x^2 + y^2 = a^2$$

– the equation of a circle.

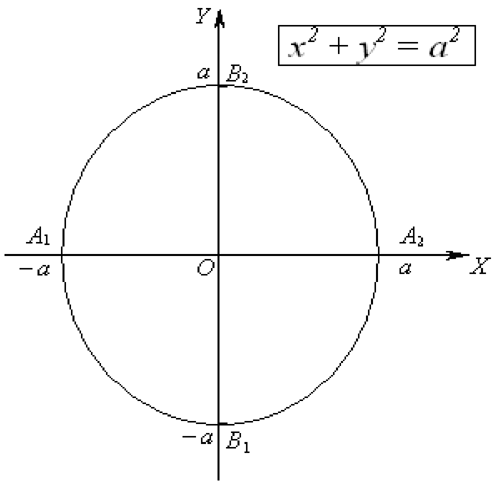


Figure 16

Eccentricity (compression)
– the ratio of the distance between the foci to the length of the major axis.
$(0 \le \varepsilon < 1)$.

1) $a > b$: $\varepsilon=\frac{c}{a}$, $\varepsilon=\sqrt{1-\left(\frac{b}{a}\right)^2}$,

2) $a < b$: $\varepsilon=\frac{c}{b}$, $\varepsilon=\sqrt{1-\left(\frac{a}{b}\right)^2}$,

3) $a = b$: $\varepsilon=0$.

Hyperbole.

A hyperbola is the locus of points in the plane for which the difference of distances to two fixed points is constant and equal $2a$. These points are called the foci. and denoted by F_1, F_2. Let $M(x, y)$ be an arbitrary point.

$|F_1F_2| = 2c,$

$\big|\,|MF_1| - |MF_2|\,\big| = 2a, \qquad (0 < a < c)$

Introduce the notation: $b = \sqrt{c^2 - a^2}$. $\quad (0 < b < c)$

The canonical equation of the hyperbole:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

a – the real semi-axis of the hyperbola,

b – the imaginary semi-axis of the hyperbola.

Construction of hyperbole.

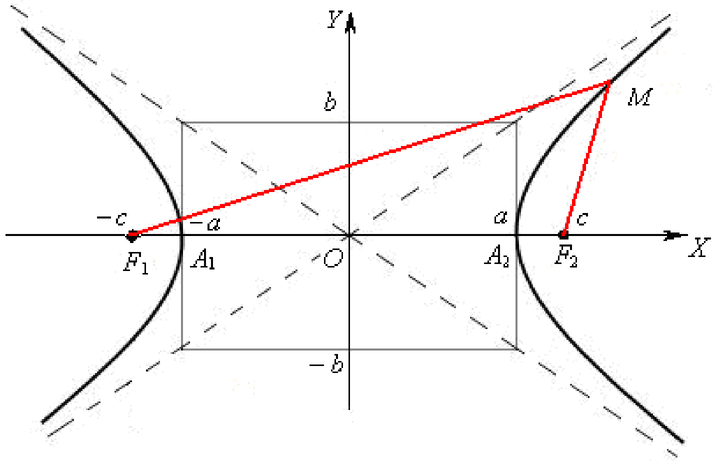


Figure 17

O – center of the hyperbole.

$A_1(-a, 0)$, $A_2(a, 0)$ – the tops of the hyperbole.

Segment $[A_1 A_2]$ and its length $2a$ – the real axis of the hyperbola. $2b$ – the imaginary axis of the hyperbola. In the considered case F_1, F_2 are on the Ox.

If F_1, F_2 are on the Oy, then the canonical equation of the hyperbole has the form:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

– the equation of a hyperbola with foci on the y-axis.

b – the real semi-axis of the hyperbola,

a – the imaginary semi-axis of the hyperbola.

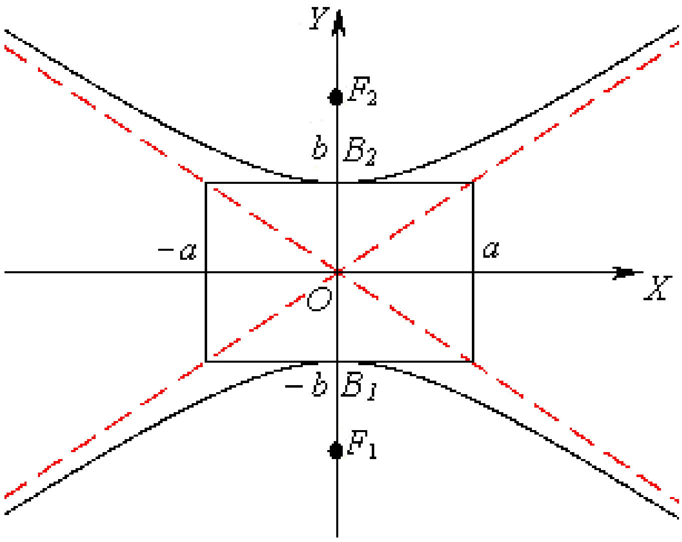


Figure 18

In this case $B_1(0, -b)$, $B_2(0, b)$ – the hyperbola tops.

If $a = b$, then we have an equilateral hyperbola.

Asymptotes.

The asymptote of a curve is a straight line approached by the curve line at infinity.

Asymptotes of hyperbola.

$$y = \pm\frac{b}{a}x.$$

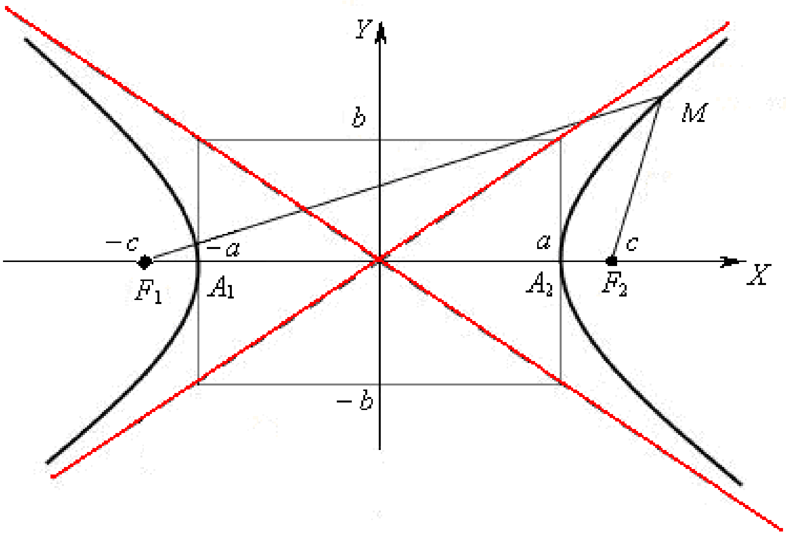


Figure 19

Eccentricity (compression)
– the ratio of the distance between the foci to the length of the real axis. ($\varepsilon > 1$)

1) a – the real semi-axis of the hyperbola: $\varepsilon = \frac{c}{a}$, $\varepsilon = \sqrt{1+\left(\frac{b}{a}\right)^2}$;

2) b – the real semi-axis of the hyperbola: $\varepsilon = \frac{c}{b}$, $\varepsilon = \sqrt{1+\left(\frac{a}{b}\right)^2}$.

Parabola.

A parabola is the locus of points in the plane, for which the distance to a fixed point equals the distance to a given straight line (a directrix).

Fixed point is called the focus.

The distance from the focus to the directrix – p ($p > 0$).

Let DD_1 – directrix, given by the equation: $x + \frac{p}{2} = 0$, focus $F\left(\frac{p}{2}, 0\right)$.

The canonical equation of a parabola:

$y^2 = 2px$, ($p > 0$), p – the parameter of the parabola.

The construction of a parabola from its equation.

O – the top of the parabola, F – the focus of the parabola, DD_1 – the directrix.

1) $y^2 = 2px$, ($p > 0$)

$O(0,0)$, $F\left(\frac{p}{2}, 0\right)$,

DD_1: $x + \frac{p}{2} = 0$;

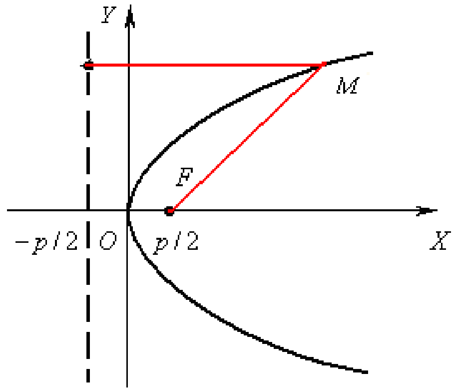


Figure 20

2) $y^2 = -2px$, ($p > 0$)

$O(0,0)$, $F\left(-\frac{p}{2}, 0\right)$,

DD_1: $x - \frac{p}{2} = 0$;

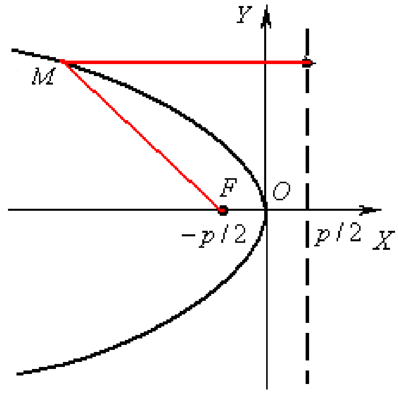


Figure 21

3) $x^2 = 2py$, ($p > 0$)

$O(0,0)$, $F\left(0, \frac{p}{2}\right)$,

DD_1: $y + \frac{p}{2} = 0$;

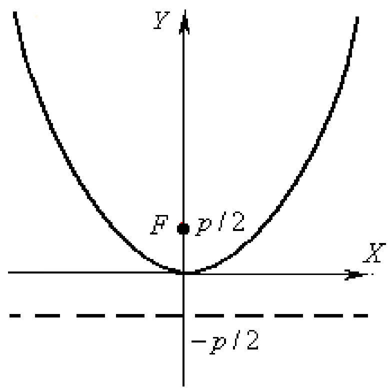


Figure 22

4) $x^2 = -2py, \quad (p > 0)$

$O(0,0), \ F\left(0, -\frac{p}{2}\right),$

$DD_1: \quad y - \frac{p}{2} = 0.$

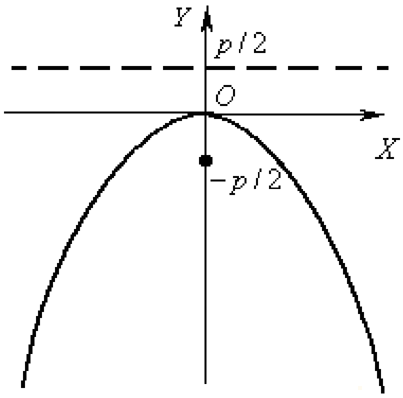


Figure 23

The eccentricity of the parabola $\varepsilon = 1$.

2 Differential calculus of one variable functions

2.1 Lecture 6. The limit of a function

Content of the lecture: Limit of function. Properties of functions which have a limit. Infinitesimal and infinitely large functions and its properties. Connection between infinitesimal and infinitely large functions. Equivalent infinitesimals and infinitely larges, using them when calculating limits.

Aims of the lecture: study the basic concepts of the theory of limits and the technique of limits calculating.

Suppose that $y = f(x)$ is a function defined in a domain D containing a point a: $a \in D$.

Definition 1 (the limit of a function at a point).

A number b is called the limit of the function $f(x)$ as x goes to a ($x \to a$) if, for any given $\varepsilon > 0$ there exists a small positive δ depending on ε ($\delta(\varepsilon) > 0$) such that, for any x satisfying the inequality $|x - a| < \delta(\varepsilon)$,

$$\left| f(x) - b \right| < \varepsilon.$$

Notation: $\lim\limits_{x \to a} f(x) = b.$

$\forall$ – the universal quantifier, $\to$ – arrow,

$\exists$ – the existential quantifier.

$$\lim_{x \to a} f(x) = b \Leftrightarrow \ \forall \varepsilon > 0 \ \exists \ \delta(\varepsilon) > 0: \ |x - a| < \delta(\varepsilon) \Rightarrow \left| f(x) - b \right| < \varepsilon.$$

Example.

1) $\lim\limits_{x \to 2}(3x + 1) = 7$ ($x = 2$ is substituted)

One-sided limits:

The left limit of the function $f(x)$ as $x \to a$ is the limit of $f(x)$ as $x \to a$ and $x < a$. Notation: $\lim\limits_{x \to a-0} f(x) = b_1$, $\quad x \to a - 0 \Leftrightarrow (x \to a, x < a)$.

The right limit of the function $f(x)$ as $x \to a$ is the limit of $f(x)$ as $x \to a$ and $x > a$. Notation: $\lim\limits_{x \to a+0} f(x) = b_2$ $\quad x \to a+0 \Leftrightarrow (x \to a, x > a)$

Remark.

1) $b_1 = b_2 = b \Rightarrow \quad b = \lim\limits_{x \to a} f(x).$

2) The right and left limits may not coincide.

Example. $\lim\limits_{x \to 0-0} \frac{1}{x} = \frac{1}{-0} = -\infty, \; \lim\limits_{x \to 0+0} \frac{1}{x} = \frac{1}{+0} = +\infty.$

Definition 2 (infinite limits).

A function $y = f(x)$ tends to infinity $(y \to \infty)$ as $x \to a$, if $\forall M > 0 \quad \exists\, \delta(M) > 0$:
for any x satisfying the inequality $|x - a| < \delta(M)$, $\quad |f(x)| > M.$

Notation: $\lim\limits_{x \to a} f(x) = \infty$.

Definition 3 (the limit of a function at infinity).

A number b is called the limit of the function $f(x)$ as $x \to \infty$ if, for any given $\varepsilon > 0$ there exists a large number N depending on ε $(N(\varepsilon) > 0)$ such that
$|f(x) - b| < \varepsilon$ for any x satisfying the inequality $|x| > N.$

Notations: $\lim\limits_{x \to \infty} f(x) = b; \quad \lim\limits_{x \to -\infty} f(x) = b; \quad \lim\limits_{x \to +\infty} f(x) = b.$

Examples.

1) $\lim\limits_{x \to \infty} \frac{x+1}{x} = \lim\limits_{x \to \infty}\left(1 + \frac{1}{x}\right) = 1;$ $\qquad$ 2) $\lim\limits_{x \to \infty} x^2 = +\infty;$ $\qquad$ 3) $\lim\limits_{x \to -\infty} x^3 = -\infty.$

Remark.

A function may not have a limit as $x \to a$ or $x \to \infty$.

Example. $\quad y = \sin x.$

Infinitesimals and infinite quantity functions.

Definition 1.

The function $\alpha(x)$ is called an infinitesimal as $x \to a$ (a – a real number or a symbol ∞), if $\lim\limits_{x \to a} \alpha(x) = 0.$

Similarly we define infinitesimal function as $x \to a - 0$ and $x \to a + 0$, as well as $x \to -\infty$ and $x \to +\infty$.

Remark.

If $\lim\limits_{x \to a} f(x) = A,$ then $(f(x) - A)$ is infinitesimal.

Definition 2.

The function $f(x)$ is called infinite quantity as $x \to a$ (a – a real number or a symbol ∞), if $\lim\limits_{x \to a} f(x) = \infty$.

Theorem 1.

1) If $f(x) \to \infty$ as $x \to a$, then $\dfrac{1}{f(x)} \to 0$ as $x \to a$;

2) if $\alpha(x) \to 0$ as $x \to a$, then $\dfrac{1}{\alpha(x)} \to \infty$ as $x \to a$.

Theorem 2.

The sum of a finitely many infinitesimals is an infinitesimal.

Theorem 3.

The product $\alpha(x) \cdot z(x)$ of an infinitesimal $\alpha(x)$ by a bounded function $z(x)$ as $x \to a$ is an infinitesimal.

Theorem 4.

The product of a finitely many infinitesimals is an infinitesimal.

Equivalent infinitesimals and their use in the calculation of the limit.

Suppose that $\alpha(x)$ and $\beta(x)$ are infinitesimals as $x \to a$.

If $\lim\limits_{x \to a} \dfrac{\alpha(x)}{\beta(x)} = 1$ then, they are said to be equivalent. Notation: $\alpha(x) \approx \beta(x)$.

Table 1

1	$\sin \alpha(x) \approx \alpha(x)$	5	$a^{\alpha(x)} - 1 \approx \alpha(x) \ln a$	9	$\ln(1 + \alpha(x)) \approx \alpha(x)$
2	$tg\,\alpha(x) \approx \alpha(x)$	6	$e^{\alpha(x)} - 1 \approx \alpha(x)$	10	$(1 + \alpha(x))^a - 1 \approx a \cdot \alpha(x)$
3	$\arcsin \alpha(x) \approx \alpha(x)$	7	$1 - \cos \alpha(x) \approx \dfrac{(\alpha(x))^2}{2}$	11	$\sqrt[n]{1 + \alpha(x)} - 1 \approx \dfrac{\alpha(x)}{n}$
4	$arctg\,\alpha(x) \approx \alpha(x)$	8	$\log_a (1 + \alpha(x)) \approx \dfrac{\alpha(x)}{\ln a}$		

Fundamental theorems on limits.

Suppose that $a \le \infty$.

Theorem 1. $\lim\limits_{x \to a} \big(f(x) + g(x)\big) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$.

Theorem 2. $\lim\limits_{x \to a} \big(f(x) \cdot g(x)\big) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$.

Corollary. $\lim\limits_{x \to a} \big(C \cdot f(x)\big) = C \cdot \lim\limits_{x \to a} f(x)$.

Theorem 3. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, $\lim\limits_{x \to a} g(x) \ne 0$.

Theorem 4 (the sandwich theorem).

If in a neighborhood of a point a $u(x) \le y(x) \le v(x)$ and $\lim_{x\to a} u(x) = \lim_{x\to a} v(x) = B$, then $\exists \lim_{x\to a} y(x) = B$.

Theorem 5.

If $u(x) \le v(x)$ and $\exists \lim_{x\to a} u(x)$, $\exists \lim_{x\to a} v(x)$, then $\lim_{x\to a} u(x) \le \lim_{x\to a} v(x)$.

Theorem 6.

If a function $y(x) \ge 0$ and $\lim_{x\to a} y(x) = b$, then $b \ge 0$.

Theorem 7.

If a function $y(x)$ is monotonically increases ($y\uparrow$) (decreases ($y\downarrow$)) and is bounded as $x \to a$ above (below), then $\exists \lim_{x\to\infty} y(x) = B$, where $B < \infty$.

First remarkable limit and its generalizations.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1,$$

generalizations:

1) $\lim_{x\to 0} \frac{\sin kx}{x} = k$; 2) $\lim_{x\to 0} \frac{tgx}{x} = 1$; 3) $\lim_{x\to 0} \frac{\sin \alpha x}{\sin \beta x} = \frac{\alpha}{\beta}$; 4) $\lim_{\substack{\alpha(x)\to 0\\ x\to a}} \frac{\sin \alpha(x)}{\alpha(x)} = 1$.

Second remarkable limit and its generalizations.

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e, \qquad e = 2{,}7182818284\ldots,$$

generalization: 1) $\lim_{N(x)\to\infty} \left(1 + \frac{1}{N(x)}\right)^{N(x)} = e$; 2) $\lim_{\substack{\alpha(x)\to 0\\ x\to a}} \left(1 + \alpha(x)\right)^{\frac{1}{\alpha(x)}} = e$;

Calculation of limits.

Definiteness: $\frac{0}{a} = 0$; $\frac{a}{0} = \infty$; $\frac{a}{\infty} = 0$; $\frac{\infty}{a} = \infty$; $a \cdot 0 = 0$; $a \cdot \infty = \infty$;

$$a^{+\infty}_{(a>1)} = \infty;\ a^{+\infty}_{(a<1)} = 0;\ a^0 = 1.$$

Indeterminacy: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 1^∞, 0^0, ∞^0.

Table 2

Indeterminacy	Methods of disclosure of indeterminacy
1 $\left[\frac{0}{0}\right]$	Allocate in the numerator and the denominator factor tending to 0 and cut it.
2 $\left[\frac{\infty}{\infty}\right]$	$\lim\limits_{x\to\infty}\frac{a_m x^m + a_{m-1}x^{m-1} + \ldots + a_0}{b_n x^n + b_{n-1}x^{n-1} + \ldots + b_0} = \lim\limits_{x\to\infty}\frac{P_m(x)}{Q_n(x)} = \left[\frac{\infty}{\infty}\right]$ compare the highest exponents of the numerator (m) and the denominator (n) $\lim\limits_{x\to\infty}\frac{P_m(x)}{Q_n(x)} = \begin{cases} \infty, & m > n \\ 0, & m < n \\ \frac{a_m}{b_n}, & m = n \end{cases}$, a_m , b_n – the coefficients of the highest degree.
3 $\left[\frac{0}{0}\right]$, trigonometry	The signal to the first remarkable limit and its equivalents $\sin x \sim x$, $\mathrm{tg}\, x \sim x$, $\arcsin x \sim x$, $\mathrm{arctg}\, x \sim x$ as $x \to 0$
4 $[1^\infty]$	The signal to the second remarkable limit $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$
5 $[0\cdot\infty]$	As $\infty = \frac{1}{0}$, $0 = \frac{1}{\infty}$, then this indeterminacy can be reduced to a kind of indeterminacy $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

2.2 Lecture 7. Continuity of functions. Differentiation

Content of the lecture: Continuity of a function. Properties of continuous at the point functions. Break points of the function and its classification. Properties of continuous on a segment functions; boundedness, existence of maximal and minimal values, existence of interim values.

Derivative of a function. Derivative of a composite function. Function given parametrically, its differentiation. Hyperbolic functions. Derivatives of hyperbolic functions. Differential of a function. Connection between derivative and differential. Geometrical meaning of the differential. Differential of a sum, product and quotient. Higher derivatives.

Aims of the lecture: study the concepts of continuous and discontinuous functions and classification of discontinuity points, the basic concepts of differential calculus of one variable functions.

1. Continuity of functions.

Suppose that the function $f(x)$ is defined in a neighborhood of x_0.

The function $f(x)$ is said to be continuous at point x_0 if

$$f(x_0+0)=f(x_0-0)=f(x_0).$$

It means that

$$\forall \varepsilon>0 \quad \exists\, \delta>0: \quad |x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon.$$

Basic properties of continuous functions in point.

Sum or product of any finite number of continuous functions is also a continuous function;

the same applies to the ratio of two continuous functions with the exception of those values of the independent variable at which the denominator vanishes.

Continuous function on the interval. $a \le x \le b$.

The function is continuous on the interval $[a, b]$, if a function is continuous for all values of x in this interval.

In this case, $\lim\limits_{x\to a+0} f(x)=f(a)$, $\lim\limits_{x\to b-0} f(x)=f(b)$.

Basic properties of continuous on the interval functions.

The extreme value theorem.

If a real-valued function $f(x)$ is continuous in the closed and bounded interval $[a, b]$, then $f(x)$ must attain a maximum and a minimum, each at least once. That is, there exist numbers c and d in $[a, b]$ such that:

$$\forall x \in [a,b]: \qquad f(c) \le f(x) \le f(d).$$

A related theorem is the boundedness theorem.

A continuous function $f(x)$ in the closed interval $[a, b]$ is bounded on that interval. That is, there exist real numbers m and M such that:

$$\forall x \in [a,b]: \quad m \le f(x) \le M.$$

Intermediate value theorem.

If $f(x)$ is a real-valued continuous function on the interval $[a, b]$, and M is a number between $f(a)$ and $f(b)$, then there is a $c \in [a, b]$ such that $f(c)=M$. In particular, if $f(a)$ and $f(b)$ have opposite signs, then $\exists\; x \in (a, b)$: $f(x)=0$.

The classification of the points discontinuities.

If $\exists\; f(c-0)$, $f(c+0)$ and $f(c+0) \ne f(c-0) \ne f(c)$

the difference $f(c+0)-f(c-0)$ is called a jump of function $f(x)$ (at point c).

Removable discontinuity: $f(c+0)=f(c-0)$, $f(c)$ – ?

Example. $f(x)=\dfrac{x^2-25}{x-5}$.

The point $x_0=5$ is a removable discontinuity; we supplement the function with $f(5)=10$.

Simple discontinuity: $\exists \; f(c-0),\; f(c+0) < \infty$, and $f(c+0) \neq f(c-0)$.

Discontinuity of the second type: $f(c-0) = \infty$ or $f(c+0) = \infty$ or does not exist.

Example.

Test the following functions for the discontinuity:

1) $f(x) = \begin{cases} x^2, & x \le 0 \\ x+1, & 0 < x \le 1 \\ -x, & x > 1 \end{cases}$;   2) $f(x) = 2^{\frac{4}{5-x}}$.

2. Differentiation.

Suppose that there is $y = f(x)$ on the $[a, b]$.

Notations: Δx – argument increment;

$\Delta y = f(x + \Delta x) - f(x)$ – function increment;

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

If $\exists \lim\limits_{\Delta x \to \pm 0} \frac{\Delta y}{\Delta x} < \infty$, this limit is called the derivative of function $f(x)$.

Lagrange notation: $y' = f'(x) = \lim\limits_{\Delta x \to \pm 0} \frac{\Delta y}{\Delta x}$.

Leibniz notation: $\frac{dy}{dx} = \frac{df(x)}{dx} = \lim\limits_{\Delta x \to \pm 0} \frac{\Delta y}{\Delta x}$.

The operation of finding the derivative of a function is called differentiation.

Remark. If for some value of x the derivative $f'(x)$ exists, then the function $f(x)$ is continuous in x.

The opposite is not true.

Geometric interpretation of a derivative.

The derivative $f'(x)$ is equal to the tangent of the angle α, formed by the tangent to the curve at the point $M(x, y)$ with the positive direction of the x-axis, i.e. is equal to the slope of the tangent.

The mechanical interpretation of a derivative.

Suppose that $S = f(t)$ – the law of motion,

Δt – time increment,

ΔS – distance increment,

then $\frac{\Delta S}{\Delta t}$ – average speed over the time interval of Δt to $t + \Delta t$,

$$v=\lim_{\Delta t\to 0}\frac{\Delta S}{\Delta t}$$ – speed at the moment t.

The derivative of distance with respect to time is the velocity at the moment *t*.

Main rules of differentiation.

1) $(c)'=0$, c – *const*,

2) $(cu)'=cu'$, $u=u(x)$

3) $(u_1+u_2+\ldots+u_n)'=u_1'+u_2'+\ldots+u_n'$

4) $(uv)'=u'v+uv'$

5) $\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$

6) $[f(\varphi(x))]'=f'_u(\varphi(x))\cdot\varphi'(x)$, $\quad y'_x=y'_u\cdot u'_x \quad y=f(u),\ u=\varphi(x)$.

7) $\varphi'(y)=\frac{1}{f'(x)}$, $\quad y=f(x)$, $x=\varphi(y)$

Table of basic derived functions.

1) $(x^n)'=nx^{n-1}$

2) $(a^x)'=a^x\cdot\ln a$

3) $e^x=e^x$

4) $(\log_a x)'=\frac{1}{x}\log_a e$

5) $(\ln x)'=\frac{1}{x}$

6) $(\sin x)'=\cos x$

7) $(\cos x)'=-\sin x$

8) $(tgx)'=\frac{1}{\cos^2 x}$

9) $(ctgx)'=-\frac{1}{\sin^2 x}$

10) $(\arcsin x)'=\frac{1}{\sqrt{1-x^2}}$

11) $(\arccos x)'=-\frac{1}{\sqrt{1-x^2}}$

12) $(arctgx)'=\frac{1}{1+x^2}$

13) $(arcctgx)'=-\frac{1}{1+x^2}$

14) $(shx)'=\left(\frac{e^x-e^{-x}}{2}\right)'=\left(\frac{e^x+e^{-x}}{2}\right)=chx$

15) $(ch\,x)'=sh\,x$

16) $(thx)'=\frac{1}{ch^2x}$

17) $(cthx)'=-\frac{1}{sh^2x}$

The equation of the tangent to the graph of one variable function $y=f(x)$.

$$y-y_0=f'(x_0)(x-x_0),$$

where $y_0=f(x_0)$,

$f'(x_0)$ – the slope of the tangent line to the graph of $y=f(x)$ at the point (x_0, y_0).

The equation of the normal to the graph of one variable function.

$$(y-y_0)=-\frac{1}{f'(x_0)}(x-x_0)$$

$\vec{n}=(f'(x_0),-1)$ – normal.

$-\dfrac{1}{f'(x_0)}$ – the slope of the normal to the graph of the function $y = f(x)$ at the point (x_0, y_0).

Method of logarithmic differentiation.

Suppose that there are functions $u = u(x)$, $v = v(x)$. $y = u^v$ – power-exponential function.

Take the logarithm of both parts, using the property of the logarithm:

$$\ln y = \ln u^v = v \cdot \ln u.$$

Differentiate both parts: $\dfrac{y'}{y} = (v \cdot \ln u)' = v' \cdot \ln u + v \cdot \dfrac{u'}{u}$.

Multiplying both parts by y, we obtain: $y' = u^v \left(v' \cdot \ln u + v \cdot \dfrac{u'}{u} \right)$.

Implicit function derivative.

Suppose that there is a dependence between argument x and function y given by the equation, not solvable with respect to y:

$$F(x, y) = 0, \qquad y = \varphi(x).$$

Differentiate both parts:

$$F'_x(x, y) + F'_y(x, y) \cdot y' = 0.$$

Solving this linear equation, we obtain:

$$y' = -\frac{F'_x(x, y)}{F'_y(x, y)}.$$

Example. $x^2 y + y^2 x = 27$,

$F(x, y) = x^2 y + y^2 x - 27 = 0$, $\quad 2xy + y^2 + (x^2 + 2xy)y' = 0$,

$$y' = -\frac{2xy + y^2}{x^2 + 2xy}$$

Derivative of the function given parametrically.

Suppose that there is function $y(x)$ given parametrically:

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}; \qquad t \in [\alpha, \beta].$$

Then the derivative can be obtained by the formula $y'_x = \dfrac{\psi'(t)}{\varphi'(t)}$.

Example.

$$\begin{cases} x = t^2 + 1, \\ y = t^3. \end{cases} \quad \varphi'(t) = 2t, \quad \psi'(t) = 3t^2, \quad y'_x = \frac{3t^2}{2t} = \frac{3}{2}t.$$

Differential and its application in approximate calculations.

Suppose that there is the differentiable function $y(x) = f(x)$ on interval $[a, b]$.

It follows that $y' = f'(x) = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \Rightarrow$

$$\Rightarrow \quad \frac{\Delta y}{\Delta x} = f'(x) + \alpha, \qquad \alpha \to 0 \text{ as } \Delta x \to 0 \quad \Rightarrow$$

$$\Rightarrow \quad \Delta y = f'(x)\Delta x + \alpha \Delta x.$$

Thus, the increment of the function Δy consists of two terms. The first term is the principal part of the increment of function. It is called the differential of function and is denoted by

$$dy = f'(x)\Delta x.$$

Find the differential of function $y = x$ by definition:

$$dy = dx = (x)' \cdot \Delta x \quad \text{or} \quad dx = \Delta x,$$

i.e. differential of the independent variable equals the increment of this variable.

Thus $\quad dy = f'(x)\, dx \quad$ and $\quad f'(x) = \frac{dy}{dx}$.

Geometric interpretation of a differential: differential of function $y = f(x)$ at the point x is an increment of ordinate tangent to function at this point, when x is incremented by Δx.

Since $\quad \Delta y = f'(x)\Delta x + \alpha \Delta x,$

then $\quad \Delta y \approx dy \quad \Rightarrow \quad f(x+\Delta x) - f(x) \approx f'(x)\Delta x.$

Thus we obtain the formula for approximate calculation:

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x\ .$$

Example. Calculate $(20.1)^2$ approximately.

$y = x^2$, $y' = 2x$, $\quad x = 20$, $\quad \Delta x = 0.1$, $\quad f(x+\Delta x) \approx f(x) + f'(x)\Delta x$,

$(20.1)^2 \approx 400+2\cdot 20\cdot 0.1=404.$

Properties of differentials:

1) $d(u+v) = du + dv$; $\qquad$ 2) $d(uv) = udv + vdu$; $\qquad$ 3) $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$.

The higher derivatives and the higher differentials.

Let the function $y = f(x)$ is differentiable on an interval $[a, b]$. $f'(x)$ is also a function of x. Differentiating this function, we obtain the second derivative of the function $f(x)$. The derivative of the $(n-1)$-th order derivative of the function $f(x)$ is called the n-th order derivative and is denoted by $y^{(n)} = (y^{(n-1)})'$.

The first differential of the $(n-1)$-th order differential of the function $f(x)$ is called the n-th order differential and is denoted by : $d^n y = d(d^{n-1}y) = f^{(n)}(x)\, dx^n$.

Using the differentials of different orders, we can express the derivative of any order: $f^{(n)}(x) = \dfrac{d^n y}{dx^n}$.

2.3 Lecture 8. Some theorems on differentiable functions

Content of the lecture: Rolle and Lagrange theorems. L'Hopital rule. Functions research: Condition of a function increase and decrease. Extreme points (necessary and sufficient conditions). Convexity and concavity of a function. Point of inflection. Curves asymptotes. General scheme of function research and its graph construction. Searching of maximal and minimal values of continuous on a segment function.

Aims of the lecture: study the basic theorems on differentiable functions, investigation of the function properties using derivatives.

L'Hopital rule. Fermat, Rolle and Lagrange theorems.

Let $f(x)$, $g(x)$ are continuous in the neighborhood of the point $x = a$, have continuous $f'(x)$, $g'(x)$ and $g'(x) \neq 0$.

Suppose that $\lim\limits_{x\to a} f(x) = 0,\ \lim\limits_{x\to a} g(x) = 0.$

Theorem (L'Hopital Rule) (disclosure of indeterminacy $\left[\dfrac{0}{0}\right]$)

If $\exists \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)} = b,$ then $\exists \lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ and $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = b.$

Remarks.

1. The rule applies to the case of the indeterminacy of the form $\left[\dfrac{\infty}{\infty}\right]$.
2. If after the application of l'Hopital rule indeterminacies of the form $\left[\dfrac{0}{0}\right]$ or $\left[\dfrac{\infty}{\infty}\right]$ remain, the rule can be applied again.

Some indeterminacies leading to the use of the l'Hopital rule:

Consider $f(x)\cdot g(x)$: $\lim\limits_{x\to a} f(x) = 0,\ \lim\limits_{x\to a} g(x) = \infty.$

$$0\cdot\infty \to \left[\frac{0}{0}\right]: \qquad f(x)\cdot g(x) = \frac{f(x)}{\left(\dfrac{1}{g(x)}\right)};$$

$$0\cdot\infty \to \left[\frac{\infty}{\infty}\right]: \qquad f(x)\cdot g(x) = \frac{g(x)}{\left(\dfrac{1}{f(x)}\right)}.$$

Consider $f(x)^{g(x)}$: $\lim\limits_{x\to a} f(x) = 1,\ \lim\limits_{x\to a} g(x) = \infty.$

$1^{\infty} \to \infty\cdot 0$: $\quad \ln[f(x)^{g(x)}] = g(x)\cdot \ln f(x)$.

Similarly, $\infty^0 \to 0\cdot\infty$, $\quad 0^0 \to 0\cdot\infty$.

Fermat Theorem.

If $f(x)$ is continuous in the interval (a, b), has a derivative at each point of this interval and at a point $x = c$ in the interval reaches the highest (lowest) value, then $\quad f'(c) = 0$.

Rolle theorem.

If $f(x)$ is differentiable on the interval $[a, b]$ and $f(a) = f(b) = 0$, then there exists at least one point within the interval $[a, b]$, where the derivative is equal to 0: $f'(c) = 0$, $\quad a < c < b$.

Lagrange theorem (finite increments of function).

If $f(x)$ is continuous on $[a, b]$ and differentiable at all interior points of the interval, then within $[a, b]$ there is at least one point c, $a < c < b$, that

$$f(b) - f(a) = f'(c)(b - a).$$

The geometric meaning of the theorem: there is such point c on the curve, that the tangent at this point is parallel to the chord

$$\frac{f(b) - f(a)}{b - a} = f'(c) = tg\alpha .$$

Increase and decrease of the function.

Theorem 1 (for an increasing function).

1) If the differentiable function $f(x)$ increases on the interval $[a, b]$, then

$$f'(x) \geq 0 \quad \text{on } [a, b].$$

2) If the function $f(x)$ is continuous on the interval $[a, b]$ and differentiable on the interval (a, b) and $f'(x) > 0$ on (a, b), then the function $f(x)$ increases on this interval.

Theorem 2 (for a decreasing function).

1) If the differentiable function $f(x)$ decreases on the interval $[a, b]$, then

$$f'(x) \leq 0 \quad \text{on } [a, b].$$

2) If the function $f(x)$ is continuous on the interval $[a, b]$ and differentiable on the interval (a, b) and $f'(x) < 0$ on (a, b), then the function $f(x)$ decreases on this interval.

To find the intervals of increase (decrease) function, we must:

1) Find the domain of existence.

2) Find the derivative of a function and equate it to zero. Solving this equation, we find the roots and divide the domain of existence into the monotonicity intervals by these points.

3) Determining the signs of the derivative within each interval, we find where the function increases and decreases.

Extremum of a function.

The function $f(x)$ has a local maximum at $x = x_1$ if

$$f(x_1 + \Delta x) < f(x_1) \qquad \forall\, \Delta x : \ |\Delta x| << 1.$$

The function $f(x)$ has a local minimum at $x = x_2$ if

$$f(x_2 + \Delta x) > f(x_2) \qquad \forall\, \Delta x : \ |\Delta x| << 1.$$

Extremums (extreme values) of functions – maximum and minimums of functions.

Remarks.

1. A function defined on an interval, can reach a maximum and minimum values for x, located inside of the segment.

2. We should not think that the maximum and minimum of the function are, respectively, the highest and lowest values in the considered interval.

Theorem 1 (a necessary condition for the existence of extremum).

If a differentiable function $y = f(x)$ has a maximum or a minimum at $x = x_1$, then $f'(x_1) = 0$.

The condition of the theorem is not sufficient. (Example: $y = x^3$).

Remark (the existence of extreme at points where the derivative does not exist (discontinuous)).

Example. $y = |x|$, $x = 0$.

The critical point (critical value) – the value of the argument at which the derivative equals zero or is discontinuous.

Theorem 2 (first sufficient condition for the existence of extremum).

Let $f(x)$ is continuous in an interval containing a critical point x_1, and differentiable at all points of the interval (except, perhaps, the point x_1), then

1) if $f'(x) > 0$ for $x < x_1$,
$f'(x) < 0$ for $x > x_1$, then at the point x_1 function has its maximum.

2) if $f'(x) < 0$ for $x < x_1$,
$f'(x) > 0$ for $x > x_1$, then at the point x_1 function has its minimum.

Theorem 3 (second sufficient condition for the existence of extremum).

Let $f'(x_1) = 0$; $f''(x)$ exists and is continuous in a neighborhood of x_1 . Then if $f''(x_1) < 0$, then the function has a maximum at this point; if $f''(x_1) > 0$, then the function has a minimum at this point.

The highest and the lowest values of a function on the interval.

Suppose that $y = f(x)$ is continuous on the interval $[a, b]$.

Then on this interval the function reaches its highest (the lowest) value at one of the ends of this interval, or in the internal point of this interval, which is the maximum (minimum).

$\Downarrow$

The rule for finding the highest (the lowest) value of the function on the interval $[a, b]$.

1) find all the critical points on the interval by solving the equation

$f'(x) = 0$;

2) compute values of the function at these points and at the ends of the interval;

3) comparing all the computed values, find the highest (the lowest) and it will be the highest (the lowest) value of the function on the interval.

Convexity and concavity of a curve. Inflections.

Suppose that $y = f(x)$ – differentiable function.

A curve $y = f(x)$ is called convex (concave) in the interval (a, b), if all points of this curve are below (above) its tangents in this interval.

Theorem 1. If $\forall x \in (a, b)$ $f''(x) < 0$, then the curve $y = f(x)$ is convex on the interval.

Theorem 1′. If $\forall x \in (a, b)$ $f''(x) > 0$, then the curve $y = f(x)$ is concave on the interval.

The point separating the convex part from the concave is called a point of inflection.

Remark.

At the point of inflection tangent, if it exists, intersects the curve.

Theorem 2 (Sufficient condition for existence of a point of inflection).

Let the curve defined by the equation $y = f(x)$.

If $f''(a) = 0$ or $f''(a)$ does not exist and, while passing over the point $x = a$ the second derivative $f''(x)$ changes its sign, then the point $x = a$ is an inflection.

Asymptotes to a curve.

Direct line, to which the curve is coming close at infinity, is called an asymptote.

Three types of asymptotes: vertical, slant, horizontal.

1) The vertical asymptote.

$$\lim_{x\to a+0} f(x) = \infty, \quad \lim_{x\to a-0} f(x) = \infty, \lim_{x\to a} f(x) = \infty \qquad \Leftrightarrow$$

direct line $x = a$ is a vertical asymptote $y = f(x)$.

2) The slant asymptote.

The slant asymptote to a curve $y = f(x)$ is defined by the equation $y = kx + b$, where $$k = \lim_{x\to\infty} \frac{f(x)}{x}, \qquad b = \lim_{x\to\infty}\left(f(x) - kx\right).$$

Remark. Arguments are valid for $x \to -\infty$.

3) The horizontal asymptote.

If $\lim\limits_{x\to\infty} f(x) = b,$ then $y = b$ is the horizontal asymptote.

Example. Find the asymptote of the curve $y = \dfrac{x^2 + 2x - 1}{x}$.

1) the vertical asymptote: $y \to +\infty$ as $x \to -0$,

$y \to -\infty$ as $x \to +0$.

Consequently, $x = 0$ is a vertical asymptote.

2) the slant asymptote:

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^2 + 2x - 1}{x^2} = 1;$$

$$b = \lim_{x\to\pm\infty}\left(f(x) - kx\right) = \lim_{x\to\pm\infty}\left(\frac{x^2 + 2x - 1}{x} - x\right) =$$

$$= \lim_{x\to\pm\infty}\left(\frac{x^2 + 2x - 1 - x^2}{x}\right) = \lim_{x\to\pm\infty}\left(\frac{2x - 1}{x}\right) = 2.$$

Consequently, $y = x + 2$ is a slant asymptote.

Investigation of the location of the curve and the asymptote:

ordinate difference: $$\left(\frac{x^2 + 2x - 1}{x} - (x + 2)\right) = -\frac{1}{x}.$$

Consequently, if $x < 0$ then the curve lies above the asymptote,

if $x > 0$ then the curve lies below the asymptote.

A complete investigation of the function and its graphing.

To analyze the function and to graph it, we must:

1) determine the domain;
2) determine the parity (odd parity);
3) determine the periodicity or nonperiodicity of the function;
4) find the points of intersection of the function and coordinate axes;
5) investigate the function for the continuity. Find the points of discontinuity and determine the type of discontinuity;
6) find intervals of increase and decrease of the function, investigate the function for the min and max;
7) find the intervals of convexity and concavity, inflection points;
8) find asymptotes of the curve;
9) construct the graph based on the above data.

3 Integral calculus of one variable function

3.1 Lecture 9. Integration

Content of the lecture: Antiderivative. Indefinite integral, its properties. The table of basic formulas of integration. Direct integration, integration by method of placing under the differential sign. Integration by parts and integration by means of change of variable.

Aims of the lecture: introduce the concept of an indefinite integral, study its properties and some of the rules and methods of integration.

Antiderivative.

Function $F(x)$ is called the antiderivative of $f(x)$ on $[a, b]$:

$\forall x \in [a, b] \quad F'(x) = f(x)$.

Example. $f(x) = x^2$, $F(x) = \dfrac{x^3}{3}$, $F(x) = \dfrac{x^3}{3} + 5$, $F(x) = \dfrac{x^3}{3} + C$, (C – *const*).

Theorem.

$F_1(x)$, $F_2(x)$ – antiderivatives of $f(x)$ on $[a, b]$ $\Rightarrow$ $F_1(x) - F_2(x) = C$, (C – *const*).

Corollary.

$F(x)$ – the antiderivative of $f(x) \Rightarrow F(x) + C$ is also the antiderivative of $f(x)$.

Indefinite integral, its properties.

Suppose that $F(x)$ is the antiderivative of $f(x)$, then $F(x) + C$ is the indefinite integral of $f(x)$.

Notation: $\int f(x)dx$.

By definition, $\int f(x)dx = F(x) + C$, if $F'(x) = f(x)$.

$f(x)$ – integrand,

$f(x)dx$ – element of integration,

$\int$ – the integral sign.

Thus indefinite integral is a family of functions $y = F(x) + C$. The operation of finding the antiderivative of $f(x)$ is called integration.

Theorem.

If the function $f(x)$ is continuous on $[a, b]$, then there exists an antiderivative (and thus indefinite integral)

Properties of the indefinite integral (by definition):

1) $\left(\int f(x)dx\right)' = \left(F(x) + C\right)' = f(x)$;

2) $d\left(\int f(x)dx\right) = f(x)dx$;

3) $\int dF(x) = F(x) + C$.

The main properties:

1) $\int [f_1(x) + f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx$;

2) $\int af(x)dx = a\int f(x)dx.$ a – const;

If $\int f(x)dx = F(x) + C,$ then

3) $\int f(ax)dx = \frac{1}{a}F(ax) + C;$

4) $\int f(x+b)dx = F(x+b) + C;$

5) $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C.$

Table of integrals.

1. $\int x^{\alpha}dx = \frac{x^{\alpha+1}}{\alpha+1} + C \qquad (\alpha \neq -1)$
2. $\int \frac{dx}{x} = \ln|x| + C$
3. $\int \sin x dx = -\cos x + C$
4. $\int \cos x dx = \sin x + C$
5. $\int \frac{dx}{\cos^2 x} = tgx + C$
6. $\int \frac{dx}{\sin^2 x} = -ctgx + C$
7. $\int tgx dx = -\ln|\cos x| + C$
8. $\int ctgx dx = \ln|\sin x| + C$
9. $\int e^x dx = e^x + C$
10. $\int a^x dx = \frac{a^x}{\ln a} + C$

11. $\int \frac{dx}{1+x^2} = arctgx + C$

11′. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} arctg\frac{x}{a} + C$

12. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

13. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

13′. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$

14. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

15. $\int shx dx = chx + C$

16. $\int chx dx = shx + C$

17. $\int \frac{dx}{ch^2 x} = thx + C$

18. $\int \frac{dx}{sh^2 x} = -cthx + C$

Methods of integration.

1) Direct integration.

Example. $\int x^4 dx = \frac{x^{4+1}}{4+1} + C = \frac{x^5}{5} + C$.

2) Change of variable or method of substitution.

$\int f(x)dx$ can be simplified by introducing a new variable t,

$$x = \varphi(t) \qquad (1)$$

Then $$\int f(x)dx = \int f[\varphi(t)]\varphi'(t)dt + C \qquad (2)$$

Often, instead of substitution (1) we use the inverse function $t = \psi(x)$.

The integration of functions with a linear argument.

$x = \varphi(t) = at + b \quad \Rightarrow \quad dx = \varphi'(t)\,dt = adt \quad (2) \quad \Rightarrow$

$$\int f(x)dx = \int f(at+b)adt + C \qquad \Rightarrow$$

$$\int f(at+b)dt = \frac{1}{a}\int f(x)dx + C$$ (main property 5).

Examples.

1. $\int (at+b)^m dt$ ($m \neq -1$)

$$\int (at+b)^m dt = \left| \begin{array}{l} x = at+b \\ dx = adt \\ dt = \frac{dx}{a} \end{array} \right| = \frac{1}{a}\int x^m dx =$$

$$= \frac{1}{a}\cdot\frac{x^{m+1}}{m+1} + C = \frac{1}{a}\cdot\frac{(at+b)^{m+1}}{m+1} + C .$$

2. $\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$.

3. (the tabulated integral 11′)

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} arctg\frac{x}{a} + C$$ (substitution $t = \frac{x}{a}$).

4. (the tabulated integral 13′)

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C .$$

3) Integration by introducing the function under the sign of differential.

$\rightarrow$ – taking out from the sign of differential (differentiation);

$\boxed{dy = y'\,dx}$

$\leftarrow$ – introduction under the sign of differential (integration).

Examples.

1. $\int \sin x \cos x dx = \int \sin x \cdot d(\sin x) = \frac{\sin^2 x}{2} + C$.

2. $\int \frac{\sin^3 x}{\cos^5 x} dx = \int tg^3 x \cdot \frac{1}{\cos^2 x} dx = \int tg^3 x \cdot d(tgx) = \frac{tg^4 x}{4} + C$.

3. $\int \frac{xdx}{x^4+1} = \int \frac{d(x^2)}{2((x^2)^2+1)} = \frac{1}{2} arctg(x^2) + C$.

4. $\int \frac{(\ln x)^5}{x} dx = \int (\ln x)^5 \cdot d(\ln x) = \frac{(\ln x)^6}{6} + C$.

4) Integration by parts.

Suppose that $u = u(x)$, $v = v(x)$ are the functions with continuous derivatives. Then we have the following

$$d(uv) = u\,dv + v\,du, \quad \text{or} \quad u\,dv = d(uv) - v\,du.$$

We integrate both sides of the expression

$$\int udv = \int [d(uv) - vdu] + C = \int d(uv) - \int vdu + C = uv - \int vdu + C.$$

Thus we obtain

$$\int udv = uv - \int vdu. \quad (3)$$

This is the formula of integration by parts.

Examples.

1. $\int \ln x dx = \begin{vmatrix} u = \ln x \to du = \frac{dx}{x} \\ dv = dx \to v = x \end{vmatrix} = x \ln x - \int x \frac{dx}{x} + C = x \ln x - x + C.$

2. $\int e^x x dx = \int x e^x dx = \begin{vmatrix} u = x \to du = dx \\ dv = e^x dx \to v = e^x \end{vmatrix} = x\ e^x - \int e^x dx = x\ e^x - e^x + C$.

The method shown in these examples is used in calculating the integrals of the type:

$$\int x^m \cdot e^{kx} dx; \qquad \int \ln kx \cdot x^m dx;$$

$$\int x^m \cdot \sin kx dx; \qquad \int \arcsin kx \cdot x^m dx;$$

$$\int x^m \cdot \cos kx dx; \qquad \int \arccos kx \cdot x^m dx;$$

$$\int arctgkx \cdot x^m dx;$$

$$\int arcctgkx \cdot x^m dx;$$

$$\int e^{mx} \cdot \sin kx dx; \quad \int e^{mx} \cdot \cos kx dx; \qquad m \in Z_+.$$

3.2 Lecture 10. Integration of rational, irrational and trigonometric functions

Content of the lecture: Integration of rational functions by expansion on partial fractions. Integration of elementary integrals containing trigonometric functions and irrational expressions.

Aims of the lecture: study the basic techniques and master the technique of integrating these types of functions.

Integration of functions containing quadratic polynomial.

Let us consider the following types of integrals:

$$I_1=\int\frac{dx}{ax^2+bx+c};\ I_2=\int\frac{Ax+B}{ax^2+bx+c}dx;\ I_3=\int\frac{dx}{\sqrt{ax^2+bx+c}};\ I_4=\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx.$$

1) Extract the perfect square

$$ax^2+bx+c=a\left(x^2+2x\frac{b}{2a}+\frac{b^2}{(2a)^2}-\frac{b^2}{(2a)^2}+\frac{c}{a}\right)=a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right].$$

We obtain

$$I_1=\int\frac{dx}{ax^2+bx+c}=\frac{1}{a}\int\frac{dx}{\left(x+\frac{b}{2a}\right)^2\pm p^2}$$ (the tabulated integrals 11′, 12)

where $\frac{c}{a}-\frac{b^2}{4a^2}=\pm p^2$ («+» is taken if $\frac{c}{a}-\frac{b^2}{4a^2}>0$; «–» is taken if $\frac{c}{a}-\frac{b^2}{4a^2}<0$).

2) $$I_2=\int\frac{Ax+B}{ax^2+bx+c}dx=\int\frac{\frac{A}{2a}(2ax+b)+\left(B-\frac{Ab}{2a}\right)}{ax^2+bx+c}dx=$$

$$=\frac{A}{2a}\int\frac{d(ax^2+bx+c)}{ax^2+bx+c}+\left(B-\frac{Ab}{2a}\right)I_1=$$

$$=\frac{A}{2a}\ln\left|ax^2+bx+c\right|+C+\left(B-\frac{Ab}{2a}\right)I_1.$$

3) $$I_3=\int\frac{dx}{\sqrt{ax^2+bx+c}}=\int\frac{dx}{\sqrt{a\left[\left(x+\frac{b}{2a}\right)^2\pm p^2\right]}}=$$

$a>0$: $$=\frac{1}{\sqrt{a}}\int\frac{dt}{\sqrt{\left(t^2\pm p^2\right)}}=\frac{1}{\sqrt{a}}\ln\left|t+\sqrt{t^2\pm p^2}\right|+C;$$

$a<0$: $$=\frac{1}{\sqrt{-a}}\int\frac{dt}{\sqrt{\left(p^2-t^2\right)}}=\frac{1}{\sqrt{-a}}\arcsin\frac{t}{p}+C.$$

4) $$I_4=\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx=\int\frac{\frac{A}{2a}(2ax+b)+\left(B-\frac{Ab}{2a}\right)}{\sqrt{ax^2+bx+c}}dx=$$

$$=\frac{A}{2a}\int\frac{d(ax^2+bx+c)}{\sqrt{ax^2+bx+c}}dx+\left(B-\frac{Ab}{2a}\right)I_3=$$

$$=\frac{A}{2a}2\sqrt{ax^2+bx+c}+C+\left(B-\frac{Ab}{2a}\right)I_3.$$

Decomposition of proper rational function in partial fractions.

The fraction $\frac{P_m(x)}{f_n(x)}=\frac{a_0x^m+a_1x^{m-1}+...+a_m}{b_0x^n+b_1x^{n-1}+...+b_n}$ is called a proper function if $m<n$; and is called an improper function if $m\geq n$.

We shall consider proper rational functions, as the improper can be reduced to proper, dividing the polynomial using long division:

$$\frac{P_m(x)}{f_n(x)}=M(x)+\frac{F_k(x)}{f_n(x)},$$

where $M(x)$ is a polynomial, $\frac{F_k(x)}{f_n(x)}=\frac{F(x)}{f(x)}$ is a proper function.

The simple fraction is a proper rational function of the form:

1) $\frac{A}{x-a}$;

2) $\dfrac{A}{(x-a)^k}$ $(k \in Z_+, k \geq 2)$;

3) $\dfrac{Ax+B}{x^2+px+q}$ (roots of the denominator are complex numbers, i.e. $p^2 - 4q < 0$);

4) $\dfrac{Ax+B}{(x^2+px+q)^k}$ (roots of the denominator are complex numbers, i.e. $p^2 - 4q < 0,\ \ k \in Z_+, k \geq 2)$.

Remark. Any proper fraction can be decomposed into a sum of simple fractions.

1-st case:

$f(x) = (x-a)(x-b)...(x-d)$.

Roots of the denominator (a, b, …, d) are different real numbers.

$$\frac{F(x)}{f(x)} = \frac{A}{x-a} + \frac{B}{x-b} + ... + \frac{D}{x-d}.$$

2-nd case:

$f(x) = (x-a)^{\alpha}(x-b)^{\beta}...(x-d)^{\delta}$.

Multiple roots of the denominator (a, b, …, d) are real numbers.

$$\frac{F(x)}{f(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + ... + \frac{A_\alpha}{(x-a)^\alpha} + \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2} + ... + \frac{B_\beta}{(x-b)^\beta} +$$

$$+ \frac{D_1}{x-d} + \frac{D_2}{(x-d)^2} + ... + \frac{D_\delta}{(x-d)^\delta}. \qquad (*)$$

3-rd case:

$f(x) = (x-a)^{\alpha}...(x-d)^{\delta}(x^2+px+q)...(x^2+lx+s)$

$$\frac{F(x)}{f(x)} = (*) + \frac{Px+Q}{x^2+px+q} + ... + \frac{Lx+S}{x^2+lx+s}.$$

4-th case:

$f(x) = (x-a)^{\alpha}...(x-d)^{\delta}(x^2+px+q)^{\mu}...(x^2+lx+s)^{\nu}$

$$\frac{F(x)}{f(x)} = (*) + \frac{P_1x+Q_1}{x^2+px+q} + ... + \frac{P_\mu x+Q_\mu}{(x^2+px+q)^\mu} + ... +$$

$$+ \frac{L_1x+S_1}{x^2+lx+s} + ... + \frac{L_\nu x+S_\nu}{(x^2+lx+s)^\nu}.$$

Integration of proper rational function.

$$1)\int \frac{A}{x-a}dx = A\ln|x-a| + C\ ;$$

$$2)\int \frac{A}{(x-a)^k}dx = A\frac{(x-a)^{-k+1}}{-k+1} + C\ ;$$

$$3)\int \frac{Ax+B}{x^2+px+q}dx = I_2\ ,\quad (k \in Z_+, k \geq 2)\ ;$$

$4)\int \frac{Ax+B}{(x^2+px+q)^k}dx$ is integrated similarly to the integral I_2.

Integration of improper rational function.

As the improper rational function can be reduced to proper, dividing the polynomial using long division:

$$\frac{P_m(x)}{f_n(x)} = M(x) + \frac{F_k(x)}{f_n(x)},$$

so we integrate the polynomial and the proper fraction.

Integration of functions containing irrational expressions.

$a)\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)dx$ replacement: $\sqrt[n]{\frac{ax+b}{cx+d}} = t$.

$b)\int R\left(x, \sqrt{a^2-x^2}\right)dx$ replacement: $x = a \sin t$.

$c)\int R\left(x, \sqrt{a^2+x^2}\right)dx$ replacement: $x = a \operatorname{tg} t$.

$d)\int R\left(x, \sqrt{x^2-a^2}\right)dx$ replacement: $x = \frac{a}{\sin t}$.

Trigonometric substitution.

$$I = \int R\left(x, \sqrt{ax^2+bx+c}\right)dx \qquad (1)$$

if $t = x + \frac{b}{2a}$, $a = m^2$ then (1) reduces to one of the form:

$1)\int R\left(t, \sqrt{n^2-m^2t^2}\right)dt$, replacement: $t = \frac{n}{m}\sin z$;

2) $\int R\left(t, \sqrt{n^2 + m^2t^2}\right)dt$, replacement: $t = \frac{n}{m}tgz$;

3) $\int R\left(t, \sqrt{m^2t^2 - n^2}\right)dt$, replacement: $t = \frac{n}{m}\frac{1}{\sin z} = \frac{n}{m}\sec z$.

As a result we obtain $I = \int \overline{R}(\sin z, \cos z)dz$

Integration of trigonometric functions. Universal substitution.

Let us consider the integral of the form:

$$\int R(\sin x, \cos x)dx .$$

This integral can be reduced to the integration of rational functions using the universal substitution: $tg\frac{x}{2} = t$.

We express sin *x*, cos *x* in terms of the tangent of half-angle:

$$\sin x = \frac{2t}{t^2+1}, \quad \cos x = \frac{1-t^2}{t^2+1}, \quad x = 2arctgt, \quad dx = 2\frac{1}{t^2+1}dt.$$

Integration of integrals of the form: $I = \int \sin^m x \cdot \cos^n x \cdot dx$.

There are two cases:

1) at least one of the numbers (m, n) is odd (for example $n = 2p + 1$):

$$I = \int \sin^m x \cdot \cos^{2p+1} x \cdot dx = \int \sin^m x \cdot \cos^{2p} x \cdot d\sin x = \int \sin^m x \cdot (1 - \sin^2 x)^p d\sin x .$$

2) Both numbers (m, n) are even ($m = 2p$, $n = 2q$).

Using the formulas of lowering the degree:

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x, \qquad \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x,$$

we obtain

$$I = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^p \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^q \cdot dx \rightarrow$$

→ decrease in the degree →even or odd degree

↓

decrease in the degree

↓

$$\int \cos kx \cdot dx .$$

Integration of functions of the form: sin *mx* · sin *nx*, sin *mx* · cos *nx*, cos *mx* · cos *nx.*

The integration of these functions can be performed by decomposition on the terms using the following formulas:

$$\sin mx \cdot \sin nx = \frac{1}{2}\left(\cos(m-n)x - \cos(m+n)x\right);$$

$$\sin mx \cdot \cos nx = \frac{1}{2}\left(\sin(m-n)x + \sin(m+n)x\right);$$

$$\cos mx \cdot \cos nx = \frac{1}{2}\left(\cos(m-n)x + \cos(m+n)x\right).$$

3.3 Lecture 11. Definite integral

Content of the lecture: Definite integral as a limit of integral sums. Basic properties of definite integral. Newton-Leibnitz formula.

Aims of the lecture: study the basic techniques and master the technique of integration of functions.

Integral sums.

Consider the continuous function $y = f(x)$ on the interval $[a, b]$.

Suppose that m and M are the smallest and the largest values of the function $f(x)$ on the interval $[a, b]$.

We divide $[a, b]$ into n parts: $a = x_0 < x_1 < x_2 < \ldots < x_n = b$,

Let us denote $x_1 - x_0 = \Delta x_1$,

$x_2 - x_1 = \Delta x_2$,

………………

$x_n - x_{n-1} = \Delta x_n$.

Denote the smallest and largest values of the function $f(x)$

on $[x_0, x_1]$ by m_1 and M_1,

on $[x_1, x_2]$ by m_2 and M_2,

…………………………………

on $[x_{n-1}, x_n]$ by m_n and M_n.

Form the sums:

1) the lower integral sum

$$\underline{s}_n = m_1\Delta x_1 + m_2\Delta x_2 + \ldots + m_n\Delta x_n = \sum_{i=1}^{n} m_i \Delta x_i \qquad (1)$$

2) the upper integral sum

$$\overline{s}_n = M_1\Delta x_1 + M_2\Delta x_2 + \ldots + M_n\Delta x_n = \sum_{i=1}^{n} M_i \Delta x_i \qquad (2)$$

The geometric meaning of the integral sums ($f(x) \geq 0$).

The properties of the lower and the upper integral sums:

a) $\underline{s}_n \le \overline{s}_n$; (3)

b) $\underline{s}_n \ge m(b-a)$; (4)

c) $\overline{s}_n \le M(b-a)$ (5)

(3)–(5) $\Rightarrow$ $m(b-a) \le \underline{s}_n \le \overline{s}_n \le M(b-a)$ (6)

(The geometric meaning if $f(x) \ge 0$).

Let us take $\xi_1, \xi_2, \ldots, \xi_n$: $x_0 < \xi_1 < x_1$,

$x_1 < \xi_2 < x_2$,

………………

$x_{n-1} < \xi_n < x_n$.

$\xi_1, \xi_2, \ldots, \xi_n \to f(\xi_1), f(\xi_2), \ldots, f(\xi_n)$.

the integral sum for $f(x)$ on $[a, b]$:

$$s_n = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \ldots + f(\xi_n)\Delta x_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

As $\forall \xi_i \in [x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$ $m_i \le f(\xi_i) \le M_i$,

then $m_i \Delta x_i \le f(\xi_i)\Delta x_i \le M_i \Delta x_i$,

therefore $\underline{s}_n \le s_n \le \overline{s}_n$.

(The geometric meaning if $f(x) \ge 0$).

We denote by $\max \Delta x_i$ – the greatest length of the segments

$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$.

Note that $\max \Delta x_i \to 0 \Rightarrow n \to \infty$.

Suppose that $s_n \to s$: $s = \lim\limits_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$.

If for any partition of the interval $[a, b]$ such that $\max \Delta x_i \to 0$, and for any choice of points ξ_i on the segments $[x_{i-1}, x_i]$ the integral sum

$$s_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i \quad (7)$$

tends to the same limit s, then this limit is called the definite integral of the function $f(x)$ on the interval $[a, b]$ and is denoted by $\int_a^b f(x)dx$.

Thus, by definition
$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx \quad (8)$$

a – the lower limit of integration,
b – the upper limit of integration.
$[a, b]$ – the interval of integration,
x – the variable of integration.

If for the function $f(x)$ the limit (8) exists, then the function is called integrable on the interval $[a, b]$.

(The geometric meaning if $f(x) \geq 0$): $$Q = \int_a^b f(x)dx.$$

Q – the area of the curvilinear trapezoid bounded by the curve $y = f(x)$, straight lines $x = a$, $x = b$ and the x-axis.

Remark.

1) $\int_a^b f(x)dx = \int_a^b f(t)dt = ... = \int_a^b f(z)dz$;

2) if $b < a$ then $\int_a^b f(x)dx = -\int_b^a f(x)dx$;

3) if $a = b$ then $\int_a^a f(x)dx = 0$.

Properties of the definite integral.

1. $\int_a^b Af(x)dx = A\int_a^b f(x)dx$, $A = const.$

2. $\int_a^b [f_1(x) + f_2(x)]dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx$.

3. If on the interval $[a, b]$ ($a < b$) $f(x) \leq \varphi(x)$, then $\int_a^b f(x)dx \leq \int_a^b \varphi(x)dx$.

4. If m and M – the smallest and largest values on $[a, b]$ and $a \leq b$ then
$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

5. If $f(x)$ is a continuous function on $[a, b]$, then $\exists \xi \in [a, b]$:
$$\int_a^b f(x)dx = (b-a)f(\xi).$$

6. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, $\forall\ a, b, c \in R.$

Only if all three integral exist.

Calculation of the definite integral.

Theorem 1. If $f(x)$ – continuous function and $\Phi(x) = \int_a^x f(t)dt$, then we have the equality $\Phi'(x) = f(x)$.

Remark.

Theorem 1 implies that every continuous function has an antiderivative. ($\Phi(x)$)

Theorem 2.

If $F(x)$ is any antiderivative of the continuous function $f(x)$, then we have the formula

$$\int_a^b f(x)dx = F(b) - F(a)$$ – Newton-Leibniz formula.

Example.

$$\int_a^b x dx = \left.\frac{x^2}{2}\right|_a^b = \frac{b^2 - a^2}{2}.$$

3.4 Lecture 12. Calculation of definite integral: substitution method and integration by parts

Content of the lecture: Substitution method in definite integral. Integration by parts of the definite integral.

Aims of the lecture: study the basic techniques and master the technique of calculation of definite integral.

Substitution method in definite integral.

Theorem.

Suppose that $\int_a^b f(x)dx$ is given, $f(x)$ is continuous on $[a, b]$.

Let us substitute variable $x = \varphi(t)$.

If

1) $\varphi(\alpha) = a$, $\varphi(\beta) = b$,
2) $\varphi(t)$, $\varphi'(t)$ are continuous on $[\alpha, \beta]$,
3) $f[\varphi(t)]$ is continuous on $[\alpha, \beta]$,

then $$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$$

Example.

$$\int_0^3 \sqrt{x+1}dx = \left| \begin{matrix} t^2 = x+1, t = \sqrt{x+1} \\ 2tdt = dx \\ x = 3 \to t = 2 \\ x = 0 \to t = 1 \end{matrix} \right| = \int_1^2 t \cdot 2tdt = 2\int_1^2 t^2 dt = \left.\frac{2}{3}t^3\right|_1^2 = \frac{2}{3}(8-1) = \frac{14}{3}.$$

Integration by parts of the definite integral.

Suppose that u and v are differentiable functions.

Then $(uv)' = u'v + uv'$.

Integrating, we obtain: $$\int_a^b (uv)'dx = \int_a^b u'v dx + \int_a^b uv'dx. \qquad (1)$$

Since $\int (uv)'dx = uv + C$, then $\int_a^b (uv)'dx = uv \Big|_a^b$;

therefore (1) can be written as $$uv \Big|_a^b = \int_a^b v du + \int_a^b u dv,$$

or finally $$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du.$$

Example.

$$\int_1^2 x \cdot e^x dx = \left| \begin{matrix} u = x, & dv = e^x dx \\ du = dx, & v = e^x \end{matrix} \right| = x \cdot e^x \Big|_1^2 - \int_1^2 e^x dx =$$

$$= 2e^2 - e - e^x \Big|_1^2 = 2e^2 - e - (e^2 - e) = e^2.$$

Bibliography

1. Khasseinov K.A. Canons of Mathematics.– Moscow: Nauka, 2007.– 592 p. – Almaty, Akbar, 2011 – 592 p., 2nd edition.
2. Хасеинов К.А. Каноны математики. Учебник.– Алматы, 2003.– 686 с.
3. Айтчанова Ш.К. Англо-русско-казахский словарь-глоссарий математический терминов.– Алматы, ЖШС «TST-company», 2005.– 123 с.
4. Пискунов Н.С. Дифференциальное и интегральное исчисления. Учебник для втузов: В 2 т.– М.: Интеграл-Пресс, 2006.–Т.1– 416 с.
5. Письменный Д.Т. Конспект лекций по высшей математике. 1 часть. – М.: Рольф, 2007. – 288 с.

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